

# THE QC YAMABE PROBLEM ON NON-SPHERICAL QUATERNIONIC CONTACT MANIFOLDS

S. IVANOV AND A. PETKOV

ABSTRACT. It is shown that the qc Yamabe problem has a solution on any compact qc manifold which is non-locally qc equivalent to the standard 3-Sasakian sphere. Namely, it is proved that on a compact non-locally spherical qc manifold there exists a qc conformal qc structure with constant qc scalar curvature.

## CONTENTS

1. Introduction	1
2. Quaternionic contact manifolds	4
2.1. Quaternionic contact structures and the Biquard connection	4
2.2. Torsion and curvature	5
2.3. Conformal change of the qc structure	6
2.4. The flat model—the quaternionic Heisenberg group	7
3. QC parabolic normal coordinates	8
3.1. QC parabolic normal coordinates	9
3.2. Scalar polynomial invariants	10
4. The asymptotic expansion of the qc Yamabe functional	11
5. Explicit evaluation of constants	22
References	30

## 1. INTRODUCTION

It is well known that the sphere at infinity of a non-compact symmetric space  $M$  of rank one carries a natural Carnot-Carathéodory structure, see [26, 27]. In the real hyperbolic case one obtains the conformal class of the round metric on the sphere. In the remaining cases, each of the complex, quaternionic and octonionic hyperbolic metrics on the unit ball induces a Carnot-Carathéodory structure on the unit sphere. This defines a conformal structure on a sub-bundle of the tangent bundle of co-dimension  $\dim_{\mathbb{R}} \mathbb{K} - 1$ , where  $\mathbb{K} = \mathbb{C}, \mathbb{H}, \mathbb{O}$ . In the complex case the obtained geometry is the well studied standard CR structure on the unit sphere in complex space. In the quaternionic case one arrives at the notion of a quaternionic contact structure. The quaternionic contact (qc) structures were introduced by O. Biquard, see [4], and are modeled on the conformal boundary at infinity of the quaternionic hyperbolic space. Biquard showed that the infinite dimensional family [24] of complete quaternionic-Kähler deformations of the quaternionic hyperbolic metric have conformal infinities which provide an infinite dimensional family of examples of qc structures. Conversely, according to [4] every real analytic qc structure on a manifold  $M$  of dimension at least eleven is the conformal infinity of a unique quaternionic-Kähler metric defined in a neighborhood of  $M$ . Furthermore, [4] considered CR and qc structures as boundaries at infinity of Einstein metrics rather than only as boundaries at infinity of Kähler-Einstein and quaternionic-Kähler metrics, respectively. In fact, in [4] it was shown that in each of the three cases (complex, quaternionic, octoninoic) any small perturbation of the standard Carnot-Carathéodory structure on the boundary is the conformal infinity of an essentially

unique Einstein metric on the unit ball, which is asymptotically symmetric. In the Riemannian case the corresponding question was posed in [8] and the perturbation result was proven in [14].

There is a deep analogy between the geometry of qc manifolds and the geometry of strictly pseudoconvex CR manifolds as well as the geometry of conformal Riemannian manifolds. The qc structures, appearing as the boundaries at infinity of asymptotically hyperbolic quaternionic manifolds, generalize to the quaternion algebra the sequence of families of geometric structures that are the boundaries at infinity of real and complex asymptotically hyperbolic spaces. In the real case, these manifolds are simply conformal manifolds and in the complex case, the boundary structure is that of a CR manifold.

A natural extension of the Riemannian and the CR Yamabe problems is the quaternionic contact (qc) Yamabe problem, a particular case of which [13, 30, 15, 16] amounts to finding the best constant in the  $L^2$  Folland-Stein Sobolev-type embedding and the functions for which the equality is achieved, [9] and [10], with a complete solution on the quaternionic Heisenberg groups given in [16, 17, 18].

Following Biquard, a quaternionic contact structure (*qc structure*) on a real  $(4n+3)$ -dimensional manifold  $M$  is a codimension three distribution  $H$  (*the horizontal distribution*) locally given as the kernel of a  $\mathbb{R}^3$ -valued one-form  $\eta = (\eta_1, \eta_2, \eta_3)$ , such that, the three two-forms  $d\eta_i|_H$  are the fundamental forms of a quaternionic Hermitian structure on  $H$ . The 1-form  $\eta$  is determined up to a conformal factor and the action of  $SO(3)$  on  $\mathbb{R}^3$ , and therefore  $H$  is equipped with a conformal class  $[g]$  of quaternionic Hermitian metrics.

For a qc manifold of crucial importance is the existence of a distinguished linear connection  $\nabla$  preserving the qc structure, defined by O. Biquard in [4], and its scalar curvature  $S$ , called qc-scalar curvature. The Biquard connection plays a role similar to the Tanaka-Webster connection [31] and [29] in the CR case. A natural question coming from the conformal freedom of the qc structures is the *quaternionic contact Yamabe problem* [30, 15, 20]: *The qc Yamabe problem on a compact qc manifold  $M$  is the problem of finding a metric  $\bar{g} \in [g]$  on  $H$  for which the qc-scalar curvature is constant.*

The question reduces to the solvability of the quaternionic contact (qc) Yamabe equation

$$\mathcal{L}f := 4 \frac{Q+2}{Q-2} \Delta f - f S = -f^{2^*-1} \bar{S},$$

where  $\Delta$  is the horizontal sub-Laplacian,  $\Delta f = \text{tr}^g(\nabla^2 f)$ ,  $S$  and  $\bar{S}$  are the qc-scalar curvatures correspondingly of  $(M, \eta)$  and  $(M, \bar{\eta} = f^{4/(Q-2)}\eta)$ , where  $2^* = \frac{2Q}{Q-2}$ , with  $Q = 4n+6$ -the homogeneous dimension.

Complete solutions to the qc Yamabe equation on the 3-Sasakian sphere, and more general, on any compact 3-Sasakian manifold were found in [18]. The case of the sphere is rather important for the general solution of the qc Yamabe problem since it provides a family of "test functions" used in attacking the general case.

The qc Yamabe problem is of variational nature as we remind next (see e.g. [30, 15, 20, 21]). Given a quaternionic contact (qc) manifold  $(M, [\eta])$  with a fixed conformal class defined by a quaternionic contact form  $\eta$ , solutions to the quaternionic contact Yamabe problem are critical points of the *qc Yamabe functional*

$$\Upsilon_M(\eta) := \frac{\int_M S \text{Vol}_\eta}{\left(\int_M \text{Vol}_\eta\right)^{2/2^*}},$$

where  $\text{Vol}_\eta$  is the natural volume form, associated to the contact form  $\eta$ , and  $2^* := \frac{2n+3}{n+1}$ . The *qc Yamabe constant* is defined by

$$\lambda(M) := \lambda(M, [\eta]) = \inf_{\eta} \Upsilon_M(\eta).$$

If  $\eta$  is a fixed qc contact form one considers the functional (which is also called *qc Yamabe functional*)

$$(1.1) \quad \Upsilon_\eta(f) := \Upsilon_M(f^{2^*-2}\eta) = \frac{\int_M \left(4 \frac{Q+2}{Q-2} |\nabla f|_\eta^2 + S f^2\right) \text{Vol}_\eta}{\left(\int_M f^{2^*} \text{Vol}_\eta\right)^{2/2^*}} = \frac{\int_M \left(4 \frac{n+2}{n+1} |\nabla f|_\eta^2 + S f^2\right) \text{Vol}_\eta}{\left(\int_M f^{2^*} \text{Vol}_\eta\right)^{2/2^*}},$$

where  $0 < f \in C^\infty(M)$ ,  $\nabla f$  is the horizontal gradient of  $f$ . The qc Yamabe constant can be expressed as

$$\lambda(M) = \inf_f \Upsilon_\eta(f)$$

and the qc Yamabe equation characterizes the non-negative extremals of the qc Yamabe functional (1.1).

The main result of W. Wang [30] states that the qc Yamabe constant of a compact quaternionic contact manifold is always less or equal than that of the standard 3-Sasakian sphere,  $\lambda(M) \leq \lambda(S^{4n+3})$  and, if the constant is strictly less than that of the sphere, the qc Yamabe problem has a solution, i.e. there exists a global qc conformal transformation sending the given qc structure to a qc structure with constant qc scalar curvature.

The qc Yamabe constant on the standard unit 3-Sasakian sphere is calculated in [16, 17, 18]. It is shown [17, Theorem 1.1] that

$$\Lambda = \lambda(S^{4n+3}) = \frac{16n(n+2)2^{\frac{1}{2n+3}}}{(2n+1)^{\frac{1}{2n+3}}} \pi^{\frac{2n+2}{2n+3}}.$$

Guided by the conformal and CR cases, a natural conjecture is that the qc Yamabe constant of every compact locally non-flat qc manifold (in conformal quaternionic contact sense) is strictly less than the qc Yamabe constant of the sphere with its standard 3-Sasakian qc structure [15, 20].

The purpose of this paper is to confirm the above conjecture. Our main result is

**Theorem 1.1.** *Suppose  $M$  is a compact qc manifold of dimension  $4n+3$ . If  $M$  is not locally qc equivalent to the standard 3-Sasakian structure on the sphere  $S^{4n+3}$  then  $\lambda(M) < \Lambda$ , and thus the qc Yamabe problem can be solved on  $M$ .*

This is analogous to the result of T. Aubin [1] for the Riemannian version of the Yamabe problem: Every compact Riemannian manifold of dimension bigger than 5 which is not locally conformally flat possesses a conformal metric of constant scalar curvature and the result of D. Jerison & J. M. Lee [22] for the CR version of the Yamabe problem: Every compact CR manifold of dimension bigger than 3 which is not locally CR equivalent to the sphere possesses a conformal pseudohermitian metric of constant pseudohermitian scalar curvature. Aubin's result is limited to dimensions bigger than 5. In the remaining cases the problem was solved by R. Schoen in [28] (see also [25] as well as [2, 3] for a different approach to the Riemannian Yamabe problem). Similarly, Jerison and Lee's result is limited to dimensions bigger than 3 and in the remaining cases the problem was solved by N. Gamara in [11] and N. Gamara and R. Yacoub in [12].

*Surprisingly in our theorem for the qc case there is no dimensional restriction which is a bit different with the Riemannian and the CR cases.*

To achieve the result we follow and adapt to the qc case the steps of the Jerison & Lee's theorem [22] which solves the CR Yamabe problem on non-spherical CR manifold of dimension bigger than 3. The main idea (as in [22]) is to find a precise asymptotic expression of the qc Yamabe functional. Our efforts throughout the present paper are concentrated on the establishment of a local asymptotic expression of the qc Yamabe functional (1.1) in terms of the Yamabe invariant of the sphere and the norm of the qc conformal curvature  $W^{qc}$  introduced by Ivanov-Vassilev in [19]. The next step is to show that the term in front of  $\|W^{qc}\|^2$  in this expression is a negative constant and then apply the qc conformal flatness theorem which states that  $W^{qc} = 0$  if and only if the qc manifold is locally qc equivalent to the standard 3-Sasakian sphere [19, Theorem 1.2, Corollary 1.3].

The key idea is that the 3-Sasakian sphere possesses a one-parameter family of extremal qc contact forms that concentrate near a point. Instead of the sphere one uses as a model the quaternionic Heisenberg group  $\mathbf{G}(\mathbb{H}) = \mathbb{H}^n \times \text{Im}(\mathbb{H})$ . The Cayley transform [15] gives a qc-equivalence between  $\mathbf{G}(\mathbb{H})$  and the sphere minus a point, which allows us to think of the standard spherical 3-Sasakian form as a qc form on  $\mathbf{G}(\mathbb{H})$ .

The Heisenberg group carries a natural family of parabolic dilations: for  $s > 0$ , the map  $\delta_s(x^\alpha, t^i) = (sx^\alpha, s^2t^i)$  is a qc automorphism of  $\mathbf{G}(\mathbb{H})$ . These dilations give rise to a family of extremal qc contact forms  $\Theta^\varepsilon = \delta_{1/\varepsilon}^* \Theta$  on  $\mathbf{G}(\mathbb{H})$  which become more and more concentrated near the origin as  $\varepsilon \rightarrow 0$ . We show that the qc Yamabe functional  $\Upsilon_M$  is closely approximated by  $\Upsilon_{\mathbf{G}(\mathbb{H})}$  for contact forms supported very near the base point.

We present a precise asymptotic expression for  $\Upsilon_M(\eta^\varepsilon)$  as  $\varepsilon \rightarrow 0$  for a suitably chosen qc contact forms  $\eta^\varepsilon$ . To this end we use the intrinsic qc normal coordinates introduced by Ch. Kunkel in [23]. The main ingredient in the Kunkel's construction of these coordinates is the fact that the tangent space of a quaternionic contact manifold has a natural parabolic dilation, instead of the more common linear dilation seen in Riemannian geometry. Kunkel used these curves to define an exponential map from the tangent space at a point to the

base manifold and showed that this exponential map incorporate this parabolic structure. Using this map and a special frame at the center point Kunkel was able to construct a set of parabolic normal coordinates. Using these parabolic normal coordinates and the effect of a conformal change of qc contact structure on the curvature and torsion tensors of the Biquard connection Kunkel was able to define a function that, when used as the conformal factor, causes the symmetrized covariant derivatives of a certain tensor constructed from the curvature and torsion to vanish. More precisely, using invariance theory, Kunkel showed in [23] that the only remaining term of weight less than or equal to 4 that does not necessarily vanish at the center point is the squared norm of the qc conformal curvature,  $\|W^{qc}\|^2$ .

Using the above coordinates around any fixed point  $q \in M$  of an arbitrary qc manifold  $(M, [\eta])$  we define the "test forms"  $\eta^\varepsilon = (f^\varepsilon)^{2*-2}\eta$ , where  $\eta$  is normalized at  $q$  qc contact form and  $f^\varepsilon$  is a suitable "test function" inspired from the solution of the qc Yamabe equation on the 3-Sasakian sphere found in [17, 18] and compute an asymptotic formula for  $\Upsilon_M(\eta^\varepsilon)$  as  $\varepsilon \rightarrow 0$ .

We show in Section 5, Theorem 5.6, that the next formula holds

$$(1.2) \quad \Upsilon_M(\eta^\varepsilon) = \Upsilon_\eta(f^\varepsilon) = \Lambda(1 - c(n)\|W^{qc}(q)\|^2\varepsilon^4) + O(\varepsilon^5),$$

where  $c(n)$  is a positive dimensional constant.

Finally, we compute the exact value of the constant  $c(n)$  and show that it is strictly positive. Since,  $W^{qc}$  is identically zero precisely when  $M$  is locally qc equivalent to the sphere [19], under the hypotheses of Theorem 1.1 there is a point  $q \in M$  where  $W^{qc}(q) \neq 0$ . This implies that for  $\varepsilon$  small enough we can achieve  $\Upsilon_M(\eta^\varepsilon) < \Lambda$  and applying the main result of W. Wang [30] we prove Theorem 1.1.

### Convention 1.2.

a) We shall use the following index convention:

$a, b, c, \dots \in \{1, \dots, 4n+3\}$ ,  $\alpha, \beta, \gamma, \dots \in \{1, \dots, 4n\}$ ,  $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \dots \in \{4n+1, 4n+2, 4n+3\}$ ,  $i, j, k, \dots \in \{1, 2, 3\}$ .

b) The summation convention over repeated indices will be used (unless otherwise stated). For example,  $A_{\alpha\beta\alpha\gamma} = \sum_{\alpha=1}^{4n} A_{\alpha\beta\alpha\gamma}$  and  $B_i C^i = \sum_{i=1}^3 B_i C^i$  and s.o.

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## 2. QUATERNIONIC CONTACT MANIFOLDS

In this section we will briefly review the basic notions of quaternionic contact geometry and recall some results from [4], [15] and [19] which we will use in this paper. Since we will work with the Kunkel's qc parabolic normal coordinates, we will follow the notations in [23].

**2.1. Quaternionic contact structures and the Biquard connection.** A quaternionic contact (qc) manifold  $(M, \eta, g, \mathbb{Q})$  is a  $4n+3$ -dimensional manifold  $M$  with a codimension three distribution  $H$  locally given as the kernel of a 1-form  $\eta = (\eta^1, \eta^2, \eta^3)$  with values in  $\mathbb{R}^3$ . In addition  $H$  has an  $Sp(n)Sp(1)$  structure, that is, it is equipped with a Riemannian metric  $g$  and a rank-three bundle  $\mathbb{Q}$  consisting of endomorphisms of  $H$  locally generated by three almost complex structures  $I^1, I^2, I^3$  on  $H$  satisfying the identities of the imaginary unit quaternions,  $I^1 I^2 = -I^2 I^1 = I^3$ ,  $I^1 I^2 I^3 = -id|_H$  which are hermitian compatible with the metric  $g(I^i \cdot, I^i \cdot) = g(\cdot, \cdot)$  with the compatibility condition  $2g(I^i X, Y) = d\eta^i(X, Y)$ ,  $i = 1, 2, 3$ ,  $X, Y \in H$ .

The transformations preserving a given qc structure  $\eta$ , i.e.,  $\bar{\eta} = f\Psi\eta$  for a positive smooth function  $f$  and an  $SO(3)$  matrix  $\Psi$  with smooth functions as entries are called *quaternionic contact conformal (qc-conformal) transformations*. If the function  $f$  is constant  $\bar{\eta}$  is called qc-homothetic to  $\eta$ . The qc conformal curvature tensor  $W^{qc}$ , introduced in [19], is the obstruction for a qc structure to be locally qc conformal to the standard 3-Sasakian structure on the  $(4n+3)$ -dimensional sphere [15, 19].

A special phenomena, noted in [4], is that the contact form  $\eta$  determines the quaternionic structure and the metric on the horizontal distribution in a unique way.

On a qc manifold with a fixed metric  $g$  on  $H$  there exists a canonical connection defined first by Biquard in [4] when the dimension  $(4n+3) > 7$ , and in [7] for the 7-dimensional case. Biquard showed that there is a unique connection  $\nabla$  with torsion  $T$  and a unique supplementary subspace  $V$  to  $H$  in  $TM$ , such that:

- (i)  $\nabla$  preserves the decomposition  $H \oplus V$  and the  $Sp(n)Sp(1)$  structure on  $H$ , i.e.  $\nabla g = 0, \nabla \sigma \in \Gamma(\mathbb{Q})$  for a section  $\sigma \in \Gamma(\mathbb{Q})$ , and its torsion on  $H$  is given by  $T(X, Y) = -[X, Y]_{|V}$ ,  $X, Y \in H$ ;
- (ii) for  $R \in V$ , the endomorphism  $T(R, \cdot)_{|H}$  of  $H$  lies in  $(sp(n) \oplus sp(1))^\perp \subset gl(4n)$ ;
- (iii) the connection on  $V$  is induced by the natural identification  $\varphi$  of  $V$  with the subspace  $sp(1)$  of the endomorphisms of  $H$ , i.e.  $\nabla \varphi = 0$ .

This canonical connection is also known as the *Biquard connection*. When the dimension of  $M$  is at least eleven Biquard [4] also described the supplementary distribution  $V$ , which is (locally) generated by the so called Reeb vector fields  $\{R_1, R_2, R_3\}$  determined by

$$(2.1) \quad \eta^i(R_j) = \delta_j^i, \quad (R_i \lrcorner d\eta^i)_{|H} = 0 \quad \text{with no summation}, \quad (R_i \lrcorner d\eta^j)_{|H} = -(R_j \lrcorner d\eta^i)_{|H},$$

where  $\lrcorner$  denotes the interior multiplication. If the dimension of  $M$  is seven Duchemin shows in [7] that if we assume, in addition, the existence of Reeb vector fields as in (2.1), then the Biquard result holds. *Henceforth, by a qc structure in dimension 7 we shall mean a qc structure satisfying (2.1).*

Letting

$$\omega_i^j = R_i \lrcorner d\eta^j|_H \quad \text{we have} \quad \omega_i^j = -\omega_j^i$$

and these are the  $sp(1)$ -connection 1-forms of  $\nabla$ , i.e. the restriction to  $H$  of the connection 1-forms for  $\nabla$  on  $V$ . The isomorphism  $\varphi$  is then simply  $\varphi(R_i) = I_i$  and these are the connection 1-forms on  $\mathbb{Q}$ .

Notice that equations (2.1) are invariant under the natural  $SO(3)$  action. Using the triple of Reeb vector fields we extend the metric  $g$  on  $H$  to a Riemannian metric on  $TM$  by requiring  $span\{R_1, R_2, R_3\} = V \perp H$ . The extended Riemannian metric  $g \oplus \sum (\eta^i)^2$  as well as the Biquard connection do not depend on the action of  $SO(3)$  on  $V$ , but both change if  $\eta$  is multiplied by a conformal factor [15]. Clearly, the Biquard connection preserves the Riemannian metric on  $TM$ .

Consider a frame  $\{\xi_\alpha, R_i\}_{\alpha=1, \dots, 4n; i=1, 2, 3}$ , where  $\{\xi_\alpha\}$  is an  $Sp(n)Sp(1)$  frame for  $H$ , and  $R_i$  are the three Reeb vector fields described above. It is occasionally convenient to have a notation for the entire frame; therefore as necessary we may refer to  $R_i$  as  $\xi_{4n+i}$ . In order to have a consistent index notation we will use different letters for different ranges of indices as in Convention 1.2. For the dual basis we use  $\theta^\alpha$  and  $\eta^i$ . We note that both  $H$  and  $V$  are orientable. The horizontal bundle is orientable since it admits an  $Sp(n)Sp(1) \subset SO(4n)$  structure and  $\mathbb{Q}$  has an  $SO(3)$  structure, hence so does  $V$ . The natural volume form on  $V$  is given by  $\epsilon = \eta^1 \wedge \eta^2 \wedge \eta^3$ , and this tensor provides a handy isomorphism between  $V$  and  $\Lambda^2 V$  (or their duals). We also denote the volume form on  $H$  by  $\Omega$  and the volume form on  $TM$  as  $dv = \Omega \wedge \epsilon$ . Using the volume form  $\epsilon^{ijk}$  and the metrics on  $H$  and  $V$ , there is a convenient way to express composition of the almost complex structures and contractions of the volume form with itself,

$$\begin{aligned} I_i^\alpha \gamma I_j^\gamma \beta &= -g_{ij} \delta_\beta^\alpha + \epsilon_{ijk} I^{k\alpha} \beta; \\ \epsilon_{ijk} \epsilon^{ilm} &= \delta_j^l \delta_k^m - \delta_k^l \delta_j^m; \quad \epsilon_{ijk} \epsilon^{ijl} = 2\delta_k^l. \end{aligned}$$

**2.2. Torsion and curvature.** Since  $T(X, Y) = -[X, Y]_{|V} \in V$  for  $X, Y \in H$ , we have

$$T_{\alpha\beta}^i = -\eta^i([\xi_\alpha, \xi_\beta]) = d\eta^i(\xi_\alpha, \xi_\beta) = 2g(I^i \xi_\alpha, \xi_\beta) = -2I_{\alpha\beta}^i, \quad \text{and} \quad T^\alpha_{\beta\gamma} = 0.$$

The properties of the Biquard connection are encoded in the properties of the torsion endomorphism  $T_R = T(R, \cdot) : H \rightarrow H$ ,  $R \in V$ . Decomposing the endomorphism  $T_R \in (sp(n) + sp(1))^\perp$  into its symmetric part  $T_R^0$  and skew-symmetric part  $b_R$ ,  $T_R = T_R^0 + b_R$ . Biquard shows in [4] that the torsion  $T_R$  is completely trace-free,  $tr T_R = tr(T_R \circ I^i) = 0$ , its symmetric part has the properties  $T_{R_i}^0 I^i = -I^i T_{R_i}^0$ ,  $I^2(T_{R_2}^0)^{+-} = I^1(T_{R_1}^0)^{-+-}$ ,  $I^3(T_{R_3}^0)^{-+-} = I^2(T_{R_2}^0)^{--+}$ ,  $I^1(T_{R_1}^0)^{--+} = I^3(T_{R_3}^0)^{+-}$ , where the superscript  $+++$  means commuting with all three  $I^i$ ,  $+-$  indicates commuting with  $I^1$  and anti-commuting with the other two and etc. The skew-symmetric part can be represented as  $b_{R_i} = I^i \mu$ , where  $\mu$  is a traceless symmetric (1,1)-tensor on  $H$  which commutes with  $I^1, I^2, I^3$ . Therefore we have  $T_{R_i} = T_{R_i}^0 + I^i \mu$ . The symmetric, trace-free endomorphism on  $H$  defined by  $\tau = (T_{R_1}^0 I_1 + T_{R_2}^0 I_2 + T_{R_3}^0 I_3)$  [15] determines completely the symmetric part [19, Proposition 2.3]  $4T_{R_i}^0 = I^i \tau - \tau I^i$ . Thus  $\tau$  together with  $\mu$  are the two  $Sp(n)Sp(1)$ -invariant components of the torsion endomorphism. If  $n = 1$  then the tensor  $\mu$  vanishes identically,  $\mu = 0$ , and the torsion is a symmetric tensor,  $T_R = T_R^0$ . Consider the Casimir operator  $\mathcal{C} = \sum_{i=1}^3 I^i \otimes I^i$  one has

$\mathcal{C}^2 = 2\mathcal{C} + 3$  and  $\tau$  belongs to the eigenspace of  $\mathcal{C}$  corresponding to the eigenvalue  $-1$  while  $\mu$  belongs to the eigenspace of  $\mathcal{C}$  corresponding to the eigenvalue  $3$ ,  $\mathcal{C}\tau = -\tau$ ,  $\mathcal{C}\mu = 3\mu$ .

Further in the paper we use the index convention. For example,  $\mathcal{C}^{\alpha\gamma}{}_{\beta\delta} = I_i^\alpha{}_\beta I^{i\gamma}{}_\delta$  and [4]

$$(2.2) \quad T^i{}_{\alpha\beta} = -2I^i{}_{\alpha\beta}, \quad T^\alpha{}_{i\alpha} = 0 = T^\alpha{}_{i\beta} I^\beta{}_\alpha, \quad \text{for any } I \in \mathbb{Q}, \quad I_i^\alpha{}_\gamma \mu^\gamma{}_\beta = \mu^\alpha{}_\gamma I_i^\gamma{}_\beta.$$

We denote  $R_{abc}{}^d = \theta^d(R(\xi_a, \xi_b)\xi_c)$  for the curvature tensor of the Biguad connection. The horizontal Ricci tensor, called *qc Ricci tensor*, is  $Ric = R_{\alpha\beta} = R_{\gamma\alpha\beta}{}^\gamma$  and the qc scalar curvature is  $S = R_\alpha{}^\alpha$ . There are nine Ricci type tensors obtained by certain contractions of the curvature tensor against the almost complex structures introduced in [15, Definition 3.7, Definition 3.9] by

$$(2.3) \quad \rho_{iab} = \frac{1}{4n} R_{ab\alpha\beta} I_i^\beta{}_\alpha, \quad \zeta_{iab} = \frac{1}{4n} R_{\alpha ab\beta} I_i^\beta{}_\alpha, \quad \sigma_{iab} = \frac{1}{4n} R_{\alpha\beta ab} I_i^\beta{}_\alpha.$$

The curvature tensor  $R_{ab\alpha\beta}$  decomposes as  $R_{ab\alpha\beta} = \mathcal{R}_{ab\alpha\beta} + \rho_{iab} I^i{}_{\alpha\beta}$ , where  $\mathcal{R}_{ab\alpha\beta}$  is the  $\mathfrak{sp}(n)$ -component of  $R_{ab\alpha\beta}$  and commutes with the almost complex structures in the second pair of indices [15, Lemma 3.8].

In fact, according to [19, Theorem 3.11] the whole curvature  $R_{abcd}$  is completely determined by its horizontal part  $R_{\alpha\beta\gamma\delta}$ , the symmetric horizontal tensors  $\tau_{\alpha\beta}, \mu_{\alpha\beta}$ , the qc scalar curvature  $S$  and their covariant derivatives up to the second order.

We collect below the necessary facts from [15, Lemma 3.11, Theorem 3.12, Corollary 3.14, Theorem 4.8] and [19, Proposition 2.3, Theorem 2.4].

$$(2.4) \quad \begin{aligned} T^\alpha{}_{i\beta} &= (T^0)^\alpha{}_{i\beta} + I_i^\alpha{}_\gamma \mu^\gamma{}_\beta = \frac{1}{4} \left( \tau^\alpha{}_\gamma I_i^\gamma{}_\beta + I_i^\alpha{}_\gamma \tau^\gamma{}_\beta \right) + I_i^\alpha{}_\gamma \mu^\gamma{}_\beta; \\ T^\alpha{}_{ij} &= d\theta^\alpha(R_i, R_j); \quad T^k{}_{ij} = -\frac{S}{8n(n+2)} \epsilon_{ij}{}^k; \quad T^\alpha{}_{\beta\gamma} = 0 = T^i{}_{j\alpha}; \\ R_{\alpha\beta} &= (2n+2)\tau_{\alpha\beta} + 2(2n+5)\mu_{\alpha\beta} + \frac{S}{4n} g_{\alpha\beta}; \\ \rho_{i\alpha\beta} &= \frac{1}{2} \left( \tau_{\alpha\gamma} I_i^\gamma{}_\beta - \tau_{\gamma\beta} I_i^\gamma{}_\alpha \right) + 2\mu_{\alpha\gamma} I_i^\gamma{}_\beta - \frac{S}{8n(n+2)} I_{i\alpha\beta}; \\ \zeta_{i\alpha\beta} &= -\frac{2n+1}{4n} \tau_{\alpha\gamma} I_i^\gamma{}_\beta + \frac{1}{4n} \tau_{\gamma\beta} I_i^\gamma{}_\alpha + \frac{2n+1}{2n} \mu_{\alpha\gamma} I_i^\gamma{}_\beta + \frac{S}{16n(n+2)} I_{i\alpha\beta}; \\ \sigma_{i\alpha\beta} &= \frac{n+2}{2n} \left( \tau_{\alpha\gamma} I_i^\gamma{}_\beta - \tau_{\gamma\beta} I_i^\gamma{}_\alpha \right) - \frac{S}{8n(n+2)} I_{i\alpha\beta}; \\ 0 &= \tau_{\alpha\beta,}{}^\beta - 6\mu_{\alpha\beta,}{}^\beta - \frac{4n-1}{2} \epsilon^{ijk} T^\beta{}_{jk} I_{i\beta\alpha} - \frac{3}{16n(n+2)} S_{, \alpha}; \\ 0 &= \tau_{\alpha\beta,}{}^\beta - \frac{n+2}{2} \epsilon^{ijk} T^\beta{}_{jk} I_{i\beta\alpha} - \frac{3}{16n(n+2)} S_{, \alpha}; \\ 0 &= \tau_{\alpha\beta,}{}^\beta - 3\mu_{\alpha\beta,}{}^\beta + 2\epsilon^{ijk} T^\beta{}_{jk} I_{i\beta\alpha} - R_{\gamma i\beta}{}^\gamma I^{i\beta}{}_\alpha. \end{aligned}$$

**2.3. Conformal change of the qc structure.** In this section we recall the conformal change of a qc structure and list the necessary facts from [15]. Let  $u$  be a smooth function on a  $(4n+3)$ -dimensional qc manifold  $(M, \eta, g, \mathbb{Q})$ . Let  $\tilde{\eta}^i = e^{2u}\eta^i$ ,  $\tilde{g} = e^{2u}g$  be the conformal transformation of the qc structure  $(\eta, g)$ . Then  $d\tilde{\eta}^i = 2e^{2u}du \wedge \eta^i + e^{2u}d\eta^i$  which restricted to  $H$  gives

$$d\tilde{\eta}^i(X, Y) = e^{2u}d\eta^i(X, Y) = 2e^{2u}g(I^i X, Y) = 2\tilde{g}(I^i X, Y), \quad X, Y \in H.$$

The new Reeb vector fields given by  $\tilde{R}_i = e^{-2u} \left( R_i - I_i^\alpha{}_\beta u^\beta \xi_\alpha \right)$  determine the new globally defined supplementary space  $\tilde{V}$ . Note that nevertheless the one forms  $\eta_i$  are local the qc conformal deformation has a global nature since the distributions  $H, V$ , the bundle  $\mathbb{Q}$  and the horizontal metric  $g$  are globally defined. We set  $\tilde{\xi}_\alpha = \xi_\alpha$  and  $\tilde{\theta}^\alpha = \theta^\alpha + I_i^\alpha{}_\beta u^\beta \eta^i$ . Then  $\tilde{\theta}^\alpha(\tilde{R}_i) = 0$  and  $\tilde{\eta}^i(\tilde{\xi}_\alpha) = 0$ . Denote by  $\mathbb{P}_{-1}$  and  $\mathbb{P}_3$  the projection onto the  $(-1)$ - and the  $3$ -eigenspaces of the operator  $\mathcal{C}$ . Then the torsion and the qc scalar



curvature changes, [15, (5.5),(5.6),(5.8)], with  $h = \frac{1}{2}e^{-2u}$ , as follows

$$(2.5) \quad \tilde{\tau}_{\alpha\beta} = \tau_{\alpha\beta} + \mathbb{P}_{-1}(4u_\alpha u_\beta - 2u_{\alpha\beta}); \quad \tilde{\mu}_{\alpha\beta} = \mu_{\alpha\beta} + \mathbb{P}_3(-2u_\alpha u_\beta - u_{\alpha\beta});$$

$$(2.6) \quad \tilde{S}\tilde{g}_{\alpha\beta} = Sg_{\alpha\beta} - 16(n+1)(n+2)u_\gamma u^\gamma g_{\alpha\beta} - 8(n+2)u_\gamma{}^\gamma g_{\alpha\beta}.$$

**2.3.1. The qc conformal curvature tensor.** In conformal geometry, the obstruction to conformal flatness is the well-studied Weyl tensor, the portion of the curvature tensor that is invariant under a conformal change of the metric and its vanishing determines if a conformal manifold is locally conformally equivalent to the standard sphere. Likewise, in the CR case, the tensor which determines local CR equivalence to the CR sphere is the Chern-Moser tensor, also determined by the curvature of the Tanaka-Webster connection and the Chern-Moser theorem [6] states that its vanishing is equivalent to the local CR equivalence to the CR sphere. And just as in the conformal case, it is the key to finding the appropriate bound for the CR Yamabe invariant on a CR manifold. Something similar appears in the qc case, dubbed the quaternionic contact conformal curvature by Ivanov and Vassilev in [19]. In that paper they define a tensor  $W^{qc}$  and prove that it is the conformally invariant portion of the Biquard curvature tensor. Moreover, if the tensor  $W^{qc}$  vanishes, they prove that the qc manifold is locally qc equivalent to the quaternionic Heisenberg group and since the quaternionic Heisenberg group and the standard 3-Sasakian sphere are locally qc equivalent [19, Theorem 1.1, Theorem 1.2, Corollary 1.3], this tensor clearly plays the role of the Weyl or Chern-Moser tensors.

The qc conformal curvature is determined by the horizontal curvature and the torsion of the Biquard connection as follows [19, (4.8)]

$$(2.7) \quad \begin{aligned} W_{\alpha\beta\gamma\delta}^{qc} = & R_{\alpha\beta\gamma\delta} + g_{\alpha\gamma}L_{\beta\delta} - g_{\alpha\delta}L_{\beta\gamma} + g_{\beta\delta}L_{\alpha\gamma} - g_{\beta\gamma}L_{\alpha\delta} \\ & + I_{i\alpha\gamma}L_{\beta\rho}I^{i\rho}{}_\delta - I_{i\alpha\delta}L_{\beta\rho}I^{i\rho}{}_\gamma + I_{i\beta\delta}L_{\alpha\rho}I^{i\rho}{}_\gamma - I_{i\beta\gamma}L_{\alpha\rho}I^{i\rho}{}_\delta \\ & + \frac{1}{2}\left(I_{i\alpha\beta}L_{\gamma\rho}I^{i\rho}{}_\delta - I_{i\alpha\beta}L_{\rho\delta}I^{i\rho}{}_\gamma + I_{i\alpha\beta}L_{\rho\sigma}I_j{}^\rho{}_\gamma I_k{}^\sigma{}_\delta \epsilon^{ijk}\right) \\ & + I_{i\gamma\delta}L_{\alpha\rho}I^{i\rho}{}_\beta - I_{i\gamma\delta}L_{\rho\beta}I^{i\rho}{}_\alpha + \frac{1}{2n}L_\rho{}^\rho I_{i\alpha\beta}I^i{}_\gamma{}^\delta, \end{aligned}$$

where the tensor  $L_{\alpha\beta}$  is given by [19, (4.6)]

$$(2.8) \quad L_{\alpha\beta} = \frac{1}{2}\tau_{\alpha\beta} + \mu_{\alpha\beta} + \frac{S}{32n(n+2)}g_{\alpha\beta}.$$

Clearly the qc conformal curvature equals the horizontal curvature precisely when the tensor  $L$  vanishes.

**2.4. The flat model—the quaternionic Heisenberg group.** The basic example of a qc manifold is provided by the quaternionic Heisenberg group  $\mathbf{G}(\mathbb{H})$  on which we introduce coordinates by regarding  $\mathbf{G}(\mathbb{H}) = \mathbb{H}^n \times \text{Im}(\mathbb{H})$ ,  $(q, \omega) \in \mathbf{G}(\mathbb{H})$  so that the multiplication takes the form  $(q_0, w_0) \circ (q, w) = (q_0 + q, w_0 + w + 2\text{Im}(q_0 \cdot \bar{q}))$ . Using real coordinates  $(x^\alpha, t^i)$ , the "standard" left invariant qc contact form

$$(2.9) \quad \Theta^i = \frac{1}{2}dt^i - I^i{}_{\alpha\beta}x^\alpha dx^\beta.$$

The dual left-invariant vertical Reeb fields  $T_1, T_2, T_3$  are  $T_1 = 2\frac{\partial}{\partial t^1}$ ,  $T_2 = 2\frac{\partial}{\partial t^2}$ ,  $T_3 = 2\frac{\partial}{\partial t^3}$ .

The left-invariant horizontal 1-forms and their dual vector fields are given by

$$(2.10) \quad \Xi^\alpha = dx^\alpha, \quad X_\alpha = \frac{\partial}{\partial x^\alpha} + 2I^i{}_{\beta\alpha}x^\beta \frac{\partial}{\partial t^i}.$$

The horizontal and vertical subbundles of  $T\mathbf{G}(\mathbb{H})$  are given by  $\mathfrak{h} = \text{Span}\{X_\alpha\}$ ,  $\mathfrak{v} = \text{Span}\{T_1, T_2, T_3\}$  and  $T\mathbf{G}(\mathbb{H}) = \mathfrak{h} \oplus \mathfrak{v}$ . On  $\mathbf{G}(\mathbb{H})$ , the left-invariant flat connection is the Biquard connection, hence  $\mathbf{G}(\mathbb{H})$  is a flat qc structure. It should be noted that the latter property characterizes (locally) the qc structure  $\Theta$  by [15, Proposition 4.11], but in fact vanishing of the curvature on the horizontal space is enough because of [19, Proposition 3.2].

### 3. QC PARABOLIC NORMAL COORDINATES

In this Section we present the necessary facts from Kunkel's work [23] concerning construction of qc parabolic normal coordinates and its consequences.

The quaternionic Heisenberg group  $\mathbf{G}(\mathbb{H})$  has a family of parabolic dilations  $(x, t) \rightarrow (sx, s^2t)$  which are automorphisms for the Lie group  $\mathbf{G}(\mathbb{H})$  and also for its Lie algebra. Then its tangent space  $T\mathbf{G}(\mathbb{H}) = \mathfrak{h} \oplus \mathfrak{v}$  come equipped with a natural parabolic dilation which sends a vector  $(v, a)$  to  $(sv, s^2a)$ , for any scalar  $s$ . Consider these vectors to be based at the origin  $o \in \mathbf{G}(\mathbb{H})$ , then by moving from  $o$  in the direction of the vector  $(v, a)$  for time  $s$  we arrive to the parametrization of a parabola,  $s \rightarrow (sv, s^2a)$ . The parabola has a simple expression in terms of differential equations as  $\ddot{\gamma}_{(v,a)} = 0$ ,  $\gamma_{(v,a)}(0) = 0$ ,  $\dot{\gamma}_{(v,a)}(0) = v$ ,  $\ddot{\gamma}_{(v,a)}(0) = a$ .

Extending this notion to a qc manifold with the Biquard connection produces curves that can rightly be called parabolic geodesics, i.e. that satisfy a natural parabolic scaling  $\gamma_{(sv, s^2a)}(t) = \gamma_{(v,a)}(st)$ . By appropriately restricting the initial conditions, Kunkel showed in [23] that there is a parabolic version of the geodesic exponential map called the parabolic exponential map. Here we state Kunkel's result for a qc manifold.

**Theorem 3.1.** *[23] Theorem 3.1.] Let  $(M, g, \mathbb{Q})$  be a qc manifold with the Biquard connection  $\nabla$  and the decomposition  $TM = H \oplus V$ . Choose any  $q \in M$  and  $(X, Y) \in H_q \oplus V_q = T_qM$  be any tangent vector. Define  $\gamma_{(X,Y)}$  to be the curve beginning at  $q$  satisfying*

$$\nabla_t^2 \dot{\gamma}_{(X,Y)} = 0, \quad \gamma_{(X,Y)}(0) = q, \quad \dot{\gamma}_{(X,Y)}(0) = X, \quad \nabla_t \dot{\gamma}_{(X,Y)}(0) = Y.$$

*Then there are neighborhoods  $0 \in O \subset T_qM$  and  $q \in O_M \subset M$  so that the function  $\Psi : O \rightarrow O_M : (X, Y) \rightarrow \gamma_{(X,Y)}(1)$  is a diffeomorphism, and satisfies the parabolic scaling  $\Psi(tX, t^2Y) = \gamma_{(X,Y)}(t)$  wherever either side is defined.*

Theorem 3.1 supplies a special frame and co-frame as follows. Let  $\{R_i\}$  be an oriented orthonormal frame for  $V_q$ , and let  $\{I_i\}$  be the associated almost complex structures. Choose an orthonormal basis  $\{\xi_\alpha\}$  for  $H_q$  so that  $\xi_{4k+i+1} = I_i \xi_{4k+1}$  for  $k = 0, \dots, n-1$ . Extending these vectors to be parallel along parabolic geodesics beginning at  $q$ , one obtains a smooth local frame for  $TM = H \oplus V$ . Define the dual 1-forms  $\{\theta^\alpha, \eta^i\}$  by  $\theta^\alpha(\xi_\beta) = \delta_\beta^\alpha$ ,  $\theta^\alpha(R_i) = 0$ ,  $\eta^i(\xi_\alpha) = 0$ ,  $\eta^i(R_j) = \delta_j^i$  and extending the almost complex structures by defining  $I_i \xi_{4k+1} = \xi_{4k+i+1}$  for  $k = 0, \dots, n-1$ , one gets the Kunkel's special frame and co-frame.

Given any special frame, one defines a coordinate map on a neighborhood of  $q$  by composing the inverse of  $\Psi$  with the map  $\lambda : T_qM \rightarrow \mathbb{R}^{4n+3} : X \rightarrow (x^\alpha, t^i) = (\theta^\alpha(X), \eta^i(X))$ . These coordinates are the Kunkel's parabolic normal coordinates (called qc pseudohermitian normal coordinates in [23]).

The generator of the parabolic dilations on the quaternionic Heisenberg group

$$\delta_s : (x, t) \rightarrow (sx, s^2t) \quad \text{is the vector field} \quad P = x^\alpha \frac{\partial}{\partial x^\alpha} + 2t^i \frac{\partial}{\partial t^i}.$$

A tensor field  $\phi$  on  $\mathbf{G}(\mathbb{H})$  is said to be homogeneous of order  $m$  if  $\mathcal{L}_P \phi = m\phi$ , where  $\mathcal{L}$  is the Lie derivative. For an arbitrary tensor field  $\phi$  the symbol  $\phi_{(m)}$  denotes the part of  $\phi$  that is homogeneous of order  $m$ .

Given a qc manifold and parabolic normal coordinates centered at a point  $q \in M$  Kunkel defined the infinitesimal generator of the parabolic dilations, the vector  $P$ , in these coordinates and shows that it is given by [23], Lemma 3.4

$$(3.1) \quad P = x^\alpha \xi_\alpha + t^i R_i. \quad \text{In particular,} \quad \theta^\alpha(P) = x^\alpha, \quad \eta^i(P) = t^i, \quad \omega_a^b(P) = 0.$$

Using this result Kunkel calculated the low order homogeneous terms of the special co-frame and the connection 1-forms, namely he proves

**Proposition 3.2.** *[23] Proposition 3.5.] In parabolic normal coordinates, the low order homogeneous terms of the special co-frame and the connection 1-forms are*

- a)  $\eta_{(2)}^i = \frac{1}{2} dt^i - I_{\alpha\beta}^i x^\alpha dx^\beta$ ;  $\eta_{(3)}^i = 0$ ;  $\eta_{(m)}^i = \frac{1}{m} (t^j \omega_j^i + T_{jk}^i t^j \eta^k - 2I_{\alpha\beta}^i x^\alpha \theta^\beta)_{(m)}$ ,  $m \geq 4$ ;
- b)  $\theta_{(1)}^\alpha = dx^\alpha$ ;  $\theta_{(2)}^\alpha = 0$ ;  $\theta_{(m)}^\alpha = \frac{1}{m} (x^\beta \omega_\beta^\alpha - T_{i\gamma}^\alpha x^\gamma \eta^i + T_{i\beta}^\alpha t^i \theta^\beta + T_{ij}^\alpha t^i \eta^j)_{(m)}$ ,  $m \geq 3$ ;
- c)  $\omega_{a(1)}^b = 0$ ;  $\omega_{a(m)}^b = \frac{1}{m} (R_{\alpha\beta a}^b x^\alpha \theta^\beta + R_{\alpha j a}^b x^\alpha \eta^j - R_{\alpha j a}^b t^j \theta^\alpha + R_{i j a}^b t^i \eta^j)_{(m)}$ ,  $m \geq 2$ .



Following [23], we denote by  $\mathcal{O}_{(m)}$  those tensor fields whose Taylor expansions at  $q$  contain only terms of order greater than or equal to  $m$ . For example, from Proposition 3.2,  $\eta^i \in \mathcal{O}_{(2)}$  and  $\theta^\alpha \in \mathcal{O}_{(1)}$ .

**Proposition 3.3.** *[23] Corollary 3.6.] If we define  $X_\alpha = \partial_\alpha + 2\Gamma_{\beta\alpha}^i x^\beta \partial_i$  and  $T_i = 2\partial_i$ , then  $\xi_\alpha = X_\alpha + \mathcal{O}_{(1)}$  and  $R_i = T_i + \mathcal{O}_{(0)}$ .*

The vector fields  $\{X_1, \dots, X_{4n}, T_1, T_2, T_3\} = \{X_a\}_{a=1}^{4n+3}$  form the standard left-invariant frame on  $\mathbf{G}(\mathbb{H})$  defined in Subsection 2.4 and the given frame on  $M$  is expressed as a perturbation of them.

It is easy to check that if  $\phi \in \mathcal{O}_{(m)}$  and  $\psi \in \mathcal{O}_{(m')}$  then  $\phi \otimes \psi \in \mathcal{O}_{(m+m')}$  and we will use the following: for any index  $a$ , let  $o(a) = 1$  if  $a \leq 4n$  and  $o(a) = 2$  if  $a > 4n$ . Given a multiindex  $A = (a_1, \dots, a_r)$  we let  $\#A = r$  and  $o(A) = \sum_{s=1}^r o(a_s)$ . If we have a collection of indexed vector fields,  $X_a$ , we let  $X_A = X_{a_r} \dots X_{a_1}$ , and similarly for similar expressions.

The next facts we need from [23] are

**Proposition 3.4.** *[23] Lemma 3.8.] Let  $F$  be a smooth function defined near  $q \in M$ . Then in parabolic normal coordinates, for any non-negative integer  $m$ ,*

$$F_{(m)} = \sum_{o(A)=m} \frac{1}{(\#A)!} \left(\frac{1}{2}\right)^{o(A)-\#A} x^A (X_A F)|_q.$$

**Lemma 3.5.** *[23] Lemma 3.9] If  $\phi$  is a tensor in  $\mathcal{O}_{(m)}$ , the components of its covariant derivatives in terms of a special frame satisfy  $\phi_{A,B} = X_B \phi_A + \mathcal{O}_{(m-o(AB)+2)}$ .*

**3.1. QC parabolic normal coordinates.** Using the qc conformal properties of a qc structure Kunkel was able to determine a conformal factor helping him to normalize the parabolic normal coordinates (qc parabolic normal coordinates) in which coordinates many tensor invariants of the qc structure vanish at the origin [[23], Section 4]. We explain briefly his results which we need in finding asymptotic expansion of the qc Yamabe functional.

Let  $\tilde{\eta} = e^{2u}\eta$  for a smooth function  $u$ . In this subsection we denote the objects with respect to  $\tilde{\eta}$  by  $\tilde{\cdot}$ , for a object  $\cdot$  with respect to  $\eta$ . Suppose that  $u$  is of order  $m \geq 2$  with respect to the vector  $P$ . Kunkel showed that the connection 1-forms of the Biquard connection  $\tilde{\omega}$  and  $\omega$  corresponding to  $\tilde{\eta}$  and  $\eta$ , respectively, satisfy

$$\tilde{\omega}_\alpha{}^\beta = \omega_\alpha{}^\beta + \mathcal{O}_{(m)}; \quad \tilde{\omega}_i{}^j = \omega_i{}^j + \mathcal{O}_{(m)}.$$

For  $u \in \mathcal{O}_{(m)}$ , applying (2.8), (2.5) and (2.6) one gets

$$(3.2) \quad \tilde{L}_{\alpha\beta} = L_{\alpha\beta} - u_{(\alpha\beta)} + \mathcal{O}_{(m-1)},$$

where we used the common notation  $u_{(\alpha\beta)} = \frac{1}{2}(u_{\alpha\beta} + u_{\beta\alpha})$  for the symmetric part of a tensor.

Consider the operator  $A$  and the 2-tensor  $B$ , defined by

$$A_{i\alpha}{}^{j\beta} = 2\epsilon^{jk}{}_i I_{k\alpha}{}^\beta, \quad B_{ij} = R_{kl\alpha\beta} \epsilon^{kl}{}_i I_j{}^{\alpha\beta}.$$

Using the first Bianchi identity from [15, 19], Kunkel [23] shows that  $A$  is invertible and

$$(3.3) \quad \begin{aligned} \tilde{A} &= A + \mathcal{O}_{(m)}; \quad \tilde{A}^{-1} = A^{-1} + \mathcal{O}_{(m)}; \\ \tilde{T}_{\alpha j k} \tilde{\epsilon}_i{}^{jk} &= T_{\alpha j k} \epsilon_i{}^{jk} + A_{i\alpha}{}^{j\beta} u_{\beta j} + \mathcal{O}_{(m-2)}; \\ \tilde{B}_{(ij)} &= B_{(ij)} + 16nu_{(ij)} + \mathcal{O}_{(m-3)}. \end{aligned}$$

Defining the symmetric tensor  $\mathcal{Q}$  with

$$(3.4) \quad \mathcal{Q}_{\alpha\beta} = L_{\alpha\beta} + \frac{1}{8(n+2)} Sg_{\alpha\beta}; \quad \mathcal{Q}_{\alpha i} = \mathcal{Q}_{i\alpha} = -(A^{-1})_{i\alpha}{}^{j\beta} T_{\beta kl} \epsilon_j{}^{kl}; \quad \mathcal{Q}_{ij} = -\frac{1}{16n} B_{(ij)},$$

it follows from (3.2) and (3.3) that for  $u \in \mathcal{O}_{(m)}$  the symmetric tensor  $\mathcal{Q}$  changes as follows [23]

$$(3.5) \quad \tilde{\mathcal{Q}}_{\alpha\beta} = \mathcal{Q}_{\alpha\beta} - u_{(\alpha\beta)} + \Delta u g_{\alpha\beta} + \mathcal{O}_{(m-1)}; \quad \tilde{\mathcal{Q}}_{i\alpha} = \mathcal{Q}_{i\alpha} - u_{i\alpha} + \mathcal{O}_{(m-2)}, \quad \tilde{\mathcal{Q}}_{ij} = \mathcal{Q}_{ij} - u_{(ij)} + \mathcal{O}_{(m-3)}.$$

In his main theorem Kunkel [[23], Theorem 3.16] proves that for any  $q \in M$  and any  $m \geq 2$  there is  $u$  which is a homogeneous polynomial of order  $m$  in parabolic normal coordinates  $(x, t)$  such that all the symmetrized covariant derivatives of  $\tilde{\mathcal{Q}}$  with total order less than or equal to  $m$  vanish at the point  $q \in M$ ,

i.e.  $\tilde{Q}_{(ab,C)}(q) = 0$  if  $o(abC) \leq m$ . Such a parabolic normal coordinates are called qc parabolic normal coordinates.

Using this normalization for  $Q$ , (2.4) and the identities from [15] and [19] Kunkel shows that in the center  $q$  of the qc parabolic normal coordinates the Ricci tensor, scalar curvature, quaternionic contact torsion, the Ricci type tensors and many of their covariant derivatives vanish,

**Theorem 3.6** ([23], Theorem 3.17). *Let  $(M, \eta, g)$  be a qc manifold for which the symmetrized covariant derivatives of the tensor  $Q$  vanish to total order 4 at a point  $q \in M$ . Then the following curvature and torsion terms vanish at  $q$ .*

$$\begin{aligned} S, \quad \tau_{\alpha\beta}, \quad \mu_{\alpha\beta}, \quad L_{\alpha\beta}, \quad R_{\alpha\beta}, \quad \rho_{i\alpha\beta}, \quad \zeta_{i\alpha\beta}, \quad \sigma_{i\alpha\beta}, \quad T_{\alpha i\beta}, \quad T_{ijk}, \quad T_{\alpha jk}, \\ S_{,\beta}, \quad \mu_{\alpha\beta, \alpha}, \quad \tau_{\alpha\beta, \alpha}, \quad B_{ij}, \quad S_{,i}, \quad S_{,\alpha}^{\alpha}, \quad \tau_{\alpha\beta, \alpha\beta}, \quad \mu_{\alpha\beta, \alpha\beta}, \quad R_{\gamma i\beta}^{\gamma}, \quad I^{i\beta\alpha}. \end{aligned}$$

We note that Kunkel proved Theorem 3.6 for a qc manifold of dimension bigger than 7, ( $n > 1$ ) but a careful examination of his proof leads that Theorem 3.6 holds also for dimension 7.

An important consequence of Theorem 3.6 is that at the center  $q$  of the qc parabolic normal coordinates the horizontal curvature is equal to its  $\mathfrak{sp}(n)$ -component, i.e. the following curvature identities hold at  $q$ :

$$(3.6) \quad \begin{aligned} R_{\alpha\beta\gamma\delta}(q) &= -R_{\beta\alpha\gamma\delta}(q); \quad R_{\alpha\beta\gamma\delta}(q) = -R_{\alpha\beta\delta\gamma}(q); \quad R_{\alpha\beta\gamma\delta}(q) + R_{\beta\gamma\alpha\delta}(q) + R_{\gamma\alpha\beta\delta}(q) = 0; \\ R_{\alpha\beta\gamma\delta}(q) &= R_{\gamma\delta\alpha\beta}(q); \quad R_{\alpha\beta\gamma\delta}(q)I_i^{\delta\varepsilon} = R_{\alpha\beta\varepsilon\delta}(q)I_i^{\delta\gamma}. \end{aligned}$$

The first identity is clear, the second one holds since the Biquard connection preserves the metric. The third, fourth and fifth equalities are a consequence of the first Bianchi identity (see e.g. [19, (3.2)]), [19, Theorem 3.1], [15, Lemma 3.8] and the fact that  $S, \tau, \mu$  all vanish at  $q$  by Theorem 3.6.

**3.2. Scalar polynomial invariants.** We recall here the definition of the notion "weight of a tensor", which plays a central role in our further considerations. First we remind the following

**Definition 3.7.** [23] *Definition 4.1.] Suppose  $F$  is a homogeneous polynomial in  $\{x^\alpha, t^i\}$  whose coefficients are polynomial expressions in the curvature, torsion and the covariant derivatives at  $q$ . We define the weight  $w(F)$  recursively by*

- a)  $w(T_{abc,D}(q)) = o(bcD) - o(a)$ ;
- b)  $w(R_{abcd,E}(q)) = o(abcdE) - o(d) = o(abE)$ , since  $c$  and  $d$  always have the same order;
- c)  $w(F_1 F_2) = w(F_1) + w(F_2)$ ;
- d)  $w(g_{ab}(q)) = w(g^{ab}(q)) = w(I_{i\alpha\beta}(q)) = w(\epsilon_{ijk}(q)) = w(c) = 0$ , where  $c$  denotes an arbitrary constant, independent of the pseudohermitian structure;
- e)  $w(0) = m$  for all  $m$ ;
- f) if  $w(F_A) = m$  for all  $A$ , then  $w(\sum_A F_A x^A) = m$ .

The notion "weight of a tensor" is an extension of the above definition. Namely, one says that a tensor  $P$  has *weight*  $m$  and designate  $w(P) = m$ , if its components with respect to the bases  $\{X_a\}_{a=1}^{4n+3}$  and  $\{\Xi^a\}_{a=1}^{4n+3}$  have weight  $m$ . Note that an arbitrary tensor  $P$  can be decomposed in homogeneous parts and the components with respect to this bases are certain homogeneous polynomial in  $\{x^\alpha, t^i\}$ .

Using the curvature and torsion identities from [15, 19] Kunkel give in [23], Table 1] the following list of curvature and torsion terms of weight less then or equal to four:

- Weight 0:  $g_{\alpha\beta}, g_{ij}, I_{i\alpha\beta}, \epsilon_{ijk}$ ;
- Weight 1:  $T_{\alpha\beta\gamma} = 0, T_{ij\alpha} = 0$ ;
- Weight 2:  $T_{\alpha i\beta}, T_{ijk}, R_{\alpha\beta\gamma\delta}$ ;
- Weight 3:  $T_{\alpha ij}, T_{\alpha i\beta, \gamma}, R_{\alpha\beta\gamma\delta, \rho}, R_{\alpha i\beta\gamma}$ ;
- Weight 4:  $T_{\alpha ij, \beta}, T_{\alpha i\beta, \gamma\delta}, T_{\alpha i\beta, j}, R_{\alpha\beta\gamma\delta, \rho\sigma}, R_{\alpha\beta\gamma\delta, i}, R_{\alpha i\beta\gamma, \delta}, R_{ij\beta\gamma}$ ,

and shows in [23], Theorem 4.3] that at the origin of a qc parabolic normal coordinates the only tensors of weight at most four are the dimensional constants and the squared norm of the qc conformal curvature tensor (2.7), namely we have

**Theorem 3.8** ([23], Theorem 4.3). *Let  $(M, \eta, g, \mathbb{Q})$  be a qc manifold. Then, at the center  $q \in M$  of the qc parabolic normal coordinates, the only invariant scalar quantities of weight no more than 4 constructed as polynomials from the invariants listed above are constants independent of the qc structure and  $\|W^{qc}\|^2$ , in particular, all other invariant scalar terms vanish at  $q \in M$ .*

In what follows we use also the next

**Convention 3.9.**

- a) We denote by  $(\eta = (\eta_1, \eta_2, \eta_3), g)$  the qc structure normalized according to Theorem 3.6. The corresponding qc parabolic normal coordinates will be signified by  $\{x^\alpha, t^i\}$ .
- b) We shall use  $\{\xi_1, \dots, \xi_{4n}, R_1, R_2, R_3\} = \{\xi_\alpha\}_{\alpha=1}^{4n+3}$  to denote the special frame, corresponding to the contact form  $\eta$ . The (dual) co-frame will be designated by  $\{\theta^1, \dots, \theta^{4n}, \eta^1, \eta^2, \eta^3\} = \{\theta^a\}_{a=1}^{4n+3}$ .
- c) The index notations of the tensors will be used only with respect to the special frame  $\{\xi_\alpha\}_{\alpha=1}^{4n+3}$  and the special co-frame  $\{\theta^a\}_{a=1}^{4n+3}$ . For example,  $A_{\alpha\beta} = A(\xi_\alpha, \xi_\beta)$ ,  $B_{\alpha\beta}^{\tilde{\gamma}} = \theta^{\tilde{\gamma}}(B(\xi_\alpha, \xi_\beta))$  and s.o.

#### 4. THE ASYMPTOTIC EXPANSION OF THE QC YAMABE FUNCTIONAL

In order to find an asymptotic expansion of the qc Yamabe functional we prove a number of lemmas.

**Lemma 4.1.** *For the standard left-invariant frame and co-frame on  $\mathbf{G}(\mathbb{H})$  the next assertions hold:*

- a) The vector fields  $X_\alpha$  are homogeneous of order  $-1$ .
- b) The vector fields  $T_i$  are homogeneous of order  $-2$ .
- c) The 1-forms  $\Xi^\alpha$  are homogeneous of order 1.
- d) The 1-forms  $\Theta^i$  are homogeneous of order 2.

*Proof.* To check a), take the Lie derivative of  $X_\alpha$  with respect to the vector field  $P$ . For a smooth function  $f$  we calculate  $(\mathcal{L}_P X_\alpha)f = -X_\alpha f$ , i.e.  $\mathcal{L}_P X_\alpha = -X_\alpha$ . Similarly,  $\mathcal{L}_P T_i = -2T_i$  which proofs b). To check c) and d), we use the Cartan formula  $\mathcal{L}_X \omega = X \lrcorner \omega + d(X \lrcorner \omega)$  for a vector field  $X \in \Gamma(TM)$  and a differential form  $\omega \in \Omega(M)$ . Standard calculations lead to  $\mathcal{L}_P \Xi^\alpha = \Xi^\alpha$  and  $\mathcal{L}_P \Theta^i = 2\Theta^i$ . Note that the last facts are implicitly mentioned in Proposition 3.2.  $\square$

The next lemma gives an information for the homogeneous parts of certain orders of the coordinate functions of the special frame  $\{\xi_\alpha\}_{\alpha=1}^{4n+3}$  with respect to the standard left-invariant vector fields  $\{X_a\}_{a=1}^{4n+3}$  on  $\mathbf{G}(\mathbb{H})$ .

**Lemma 4.2.** *If  $\xi_\alpha = s_\alpha^b X_b$ , then the following relations hold:*

$$(4.1) \quad s_{\alpha(0)}^\beta = \delta_\alpha^\beta, \quad s_{\alpha(0)}^{\tilde{\alpha}} = s_{\alpha(1)}^{\tilde{\alpha}} = s_{\alpha(0)}^\alpha = s_{\alpha(1)}^{\tilde{\beta}} = 0, \quad s_{\alpha(0)}^{\tilde{\beta}} = \delta_\alpha^{\tilde{\beta}}.$$

Moreover, for a natural number  $m$ , the next recursive formula is true:

$$(4.2) \quad s_{a(m+o(b)-o(a))}^b = - \sum_{i \geq 2} s_{a(m+o(c)-o(a)-i)}^c \theta_{(o(b)+i)}^b(X_c).$$

*Proof.* First, we have  $\xi_\alpha = s_\alpha^\beta X_\beta + s_\alpha^{\tilde{\alpha}} X_{\tilde{\alpha}}$ . We get by Proposition 3.3 and Lemma 4.1 that  $\xi_\alpha \in \mathcal{O}_{(-1)}$ ,  $s_\alpha^\beta X_\beta \in \mathcal{O}_{(-1)}$  and  $s_\alpha^{\tilde{\alpha}} X_{\tilde{\alpha}} \in \mathcal{O}_{(-2)}$  (note that  $s_a^b \in \mathcal{O}_{(0)}$ ). Taking the homogeneous parts of order  $-1$  and  $-2$  in the above equality, we obtain that  $s_{\alpha(0)}^\beta = \delta_\alpha^\beta$ ,  $s_{\alpha(1)}^{\tilde{\alpha}} = 0$  and  $s_{\alpha(0)}^{\tilde{\alpha}} = 0$ , respectively. Similarly, taking the homogeneous parts of order  $-2$  and  $-1$  in the equality  $\xi_{\tilde{\alpha}} = s_{\tilde{\alpha}}^\alpha X_\alpha + s_{\tilde{\alpha}}^{\tilde{\beta}} X_{\tilde{\beta}}$  we get  $s_{\tilde{\alpha}(0)}^{\tilde{\beta}} = \delta_{\tilde{\alpha}}^{\tilde{\beta}}$  and  $s_{\tilde{\alpha}(0)}^\alpha = s_{\tilde{\alpha}(1)}^{\tilde{\beta}} = 0$ , respectively, which proofs (4.1).

In order to prove (4.2) we take the homogeneous parts of order  $m+o(b)-o(a)$  in the equality  $\delta_a^b = s_a^c \theta^b(X_c)$ . We separate the proof in two cases. The first case appears when  $m+o(b)-o(a) > 0$ . Then we get

$$0 = \delta_{a(m+o(b)-o(a))}^b = \sum_{i \geq 0} s_{a(m+o(c)-o(a)-i)}^c [\theta^b(X_c)]_{(o(b)-o(c)+i)}, \quad \text{i.e.}$$

$$s_{a(m+o(b)-o(a))}^b = - \sum_{i \geq 2} s_{a(m+o(c)-o(a)-i)}^c \theta_{(o(b)+i)}^b(X_c).$$

Note that the left-hand side of the last identity is just the term in the sum that corresponds to  $i = 0$ , while the term that correlates to  $i = 1$  is equal to 0, by Proposition 3.2.

The case  $m + o(b) - o(a) = 0$  occurs only when  $m = 1, o(b) = 1$  and  $o(a) = 2$ . Hence, we have  $b = \alpha, a = \tilde{\alpha}$  and the left-hand side of (4.2) becomes  $s_{\tilde{\alpha}(0)}^\alpha = 0$ , according to (4.1). Moreover,  $s_{\tilde{\alpha}(o(c)-1-i)}^c = 0$  for  $i \geq 2$ , and the right-hand side of (4.2) becomes also 0 which completes the proof.  $\square$

A crucial auxiliary result that help us to find an asymptotic expansion of the qc Yamabe functional is

**Lemma 4.3.** *The differential forms  $\eta_{(m)}^i$  and  $d\eta_{(m)}^i$  have weight  $m - 2$ . The vector fields  $\xi_{\alpha(m)}$  have weight  $m + 1$ , while the vector fields  $\xi_{\tilde{\alpha}(m)}$  have weight  $m + 2$ . Finally, the function  $S_{(m)}$  has weight  $m + 2$ .*

*Proof.* We shall prove by induction on  $k$  that all the objects

$$(4.3) \quad \begin{aligned} &\eta_{(k+2)}^i, \quad d\eta_{(k+2)}^i, \quad \theta_{(k+1)}^\alpha, \quad \omega_{a(k)}^b, \quad \xi_{a(k-o(a))}, \quad s_{a(k+o(b)-o(a))}^b, \\ &T_{abc,A(k-o(A)-o(bc)+o(a))}, \quad R_{abcd,A(k-o(A)-o(ab))} \end{aligned}$$

have weight  $k$ .

A) *Base of the induction:*  $k = 0$ . We have  $\eta_{(2)}^i = 1 \cdot \Theta^i$  by Proposition 3.2 and  $w(1) = 0$  by Definition 3.7, so  $w(\eta_{(2)}^i) = 0$ . In the same way,  $w(d\eta_{(2)}^i) = w(\theta_{(1)}^\alpha) = w(\omega_{a(0)}^b) = 0$ . Proposition 3.3 and Lemma 4.1 imply  $\xi_{a(-o(a))} = 1 \cdot X_a$  which combined with  $w(1) = 0$  yield  $w(\xi_{a(-o(a))}) = 0$ . From (4.1) we have  $s_{\alpha(o(\beta)-o(\alpha))}^\beta = s_{\alpha(0)}^\beta = \delta_\alpha^\beta$ , and similarly for the other choices of the indices  $a$  and  $b$ . Hence, we get  $w(s_{a(o(b)-o(a))}^b) = 0$ . Next, we have  $T_{abc,A(-o(A)-o(bc)+o(a))} = 0$  except the case when  $A = \emptyset, a = \tilde{\alpha}, b = \alpha, c = \beta$ , in which situation we obtain  $T_{\tilde{\alpha}\alpha\beta(0)} = -2I_{\tilde{\alpha}\alpha\beta} = \text{Const.}$  Thus,  $w(T_{abc,A(-o(A)-o(bc)+o(a))}) = 0$ . Finally, we obtain in the same manner that  $w(R_{abcd,A(-o(A)-o(ab))}) = 0$ , which completes the base of the induction.

B) *Inductive step.* Suppose that all the objects in (4.3) have weight  $k$  for  $k \leq m$ . We are going to prove it holds for  $k = m + 1$ . The first step is to check the assertion for the torsion and the curvature when  $A = \emptyset$ .

First we have to show that  $T_{abc(m+1-o(bc)+o(a))}$  has weight  $m + 1$ . Applying Proposition 3.4, we get

$$T_{abc(m+1-o(bc)+o(a))} = \sum_A \frac{1}{(\#A)!} \left(\frac{1}{2}\right)^{o(A)-\#A} x^A (X_A T_{abc})|_q,$$

where the sum is taken over all multi-indices  $A : o(A) = m + 1 - o(bc) + o(a)$ . We show that  $X_A T_{abc}|_q$  has weight  $m + 1$  for any multi-index  $A$ . (Note that we use here the same letter  $A$  to denote the corresponding multi-index; we are doing this now and later in order to avoid the excessive accumulation of letters). For that purpose, we will prove that  $\xi_A T_{abc}|_q$  has weight  $m + 1$  for any multi-index  $A$  with the mentioned order.

We recall that if  $\{T_{a_1 \dots a_r}\}$  are the components of a tensor  $T$  of type  $(0, r)$  with respect to the frame  $\{\xi_a\}_{a=1}^{4n+3}$  then the components of the covariant derivative of  $T$  along the vector field  $\xi_b$  are given by

$$(4.4) \quad T_{A,b} = \xi_b T_A - \sum_{i=1}^r \omega_{a_i}^c(\xi_b) T_{a_1 \dots a_{i-1} c a_{i+1} \dots a_r}, \quad A = (a_1 \dots a_r).$$

We introduce the notation:

$$P_{abc,Ad} := \omega_a^e(\xi_d) T_{ebc,A} + \omega_b^e(\xi_d) T_{aec,A} + \omega_c^e(\xi_d) T_{abe,A} + \sum_{i=1}^r \omega_{a_i}^e(\xi_d) T_{abc,a_1 \dots a_{i-1} e a_{i+1} \dots a_r}, \quad A = (a_1 \dots a_r).$$

Taking into account (4.4) and the above notation, it is not difficult to see that

$$(4.5) \quad T_{abc,A} = \xi_A T_{abc} - \sum_{i=1}^r \xi_{A_i} (P_{abc,B_i}),$$

where  $A_i := (a_{i+1} \dots a_r)$ ,  $i = 0, \dots, r-1$ ,  $A_r := \emptyset$ ,  $B_i := (a_1 \dots a_i)$ ,  $i = 1, \dots, r$ ,  $B_0 := \emptyset$ .

The next step is to take in (4.5) the homogeneous parts of order 0. At first, we shall prove that

$$(4.6) \quad w(P_{abc,B_i(k-o(B_i)-o(bc)+o(a))}) = k, \quad i = 1, \dots, r, \quad k \leq m + 1.$$

We have

$$(4.7) \quad [\omega_a^d(\xi_{a_i}) T_{dbc,B_{i-1}}]_{(k-o(B_i)-o(bc)+o(a))} = \sum_{k_1, k_2} [\omega_a^d(\xi_{a_i})]_{(k_1)} [T_{dbc,B_{i-1}}]_{(k_2)},$$

where  $k_1, k_2$  satisfy  $k_1 + k_2 = k - o(B_i) - o(bc) + o(a)$ . Since  $k_1, k_2 \geq 0$ , we get  $k_1 \leq k - o(B_i) - o(bc) + o(a)$  and  $k_2 \leq k - o(B_i) - o(bc) + o(a)$ , i.e. we obtain the conditions

$$(4.8) \quad k_1 \leq m + 1 - o(B_i) - o(bc) + o(a), \quad k_2 \leq m + 1 - o(B_i) - o(bc) + o(a).$$

To investigate the weight of the term  $[\omega_a^d(\xi_{a_i})]_{(k_1)}$ , we decompose it in the sum

$$(4.9) \quad [\omega_a^d(\xi_{a_i})]_{(k_1)} = \sum_{l_1, l_2} \omega_{a(l_1)}^d \xi_{a_i(l_2)},$$

where  $l_1$  and  $l_2$  satisfy  $l_1 + l_2 = k_1$ ,  $l_1 \geq 2$ ,  $l_2 \geq -o(a_i)$ . These conditions and the first inequality in (4.8) give  $l_1 = k_1 - l_2 \leq m + 1 - o(B_{i-1}) - o(bc) + o(a) \leq m$ . The last inequality does not hold only in the trivial case when  $B_{i-1} = \emptyset$ ,  $o(bc) = 2$ ,  $o(a) = 2$ . But this situation occurs only if  $a \in \{4n + 1, 4n + 2, 4n + 3\}$ ,  $b, c \in \{1, \dots, 4n\}$ , which implies  $k_1, k_2 \leq m$  and we can apply the inductive hypothesis to the terms that appears in the right-hand side of (4.7) to obtain that the term in the left-hand side of (4.7) has weight  $k$ . So, we can apply the inductive hypothesis for  $\omega_a^d$  to conclude that

$$(4.10) \quad w(\omega_{a(l_1)}^d) = l_1.$$

Moreover, the first inequality in (4.8) and the inequality  $l_1 \geq 2$  gives  $l_2 = k_1 - l_1 \leq m - 2$ , which allows to apply the inductive hypothesis for  $\xi_{a_i}$ , i.e.

$$(4.11) \quad w(\xi_{a_i(l_2)}) = l_2 + o(a_i).$$

Using (4.10), (4.11) and (4.9), we conclude

$$(4.12) \quad w\{[\omega_a^d(\xi_{a_i})]_{(k_1)}\} = k_1 + o(a_i).$$

Furthermore, the second inequality in (4.8) implies  $k_2 + o(B_{i-1}) + o(bc) - o(a) \leq m$  which together with the inductive hypothesis for the torsion yield

$$(4.13) \quad w[(T_{dbc, B_{i-1}})_{(k_2)}] = k_2 + o(B_{i-1}) + o(bc) - o(d).$$

Finally, we get from (4.7), (4.12) and (4.13) that  $[\omega_a^d(\xi_{a_i})T_{dbc, B_{i-1}}]_{(k - o(B_i) - o(bc) + o(a))}$  has weight  $k$ . We obtain in the same manner that the homogeneous parts of order  $k - o(B_i) - o(bc) + o(a)$  of the other terms in the definition of  $P_{abc, B_i}$  all have weight  $k$ , which proofs (4.6).

To calculate the weight of  $\xi_A T_{abc}|_q$  we consider  $[\xi_{A_i}(P_{abc, B_i})]_{(0)}$ ,  $i = 1, \dots, r$ , with the decomposition:

$$(4.14) \quad \begin{aligned} [\xi_{A_i}(P_{abc, B_i})]_{(0)} &= \sum_{k_0 + k_{i+1} + \dots + k_r = 0} \xi_{a_r(k_r)} \dots \xi_{a_{i+1}(k_{i+1})} (P_{abc, B_i})_{(k_0)} \\ &= \sum_{k_0 + k_{i+1} + \dots + k_r = 0} \xi_{a_r(k_r + o(a_r) - o(a_r))} \dots \xi_{a_{i+1}(k_{i+1} + o(a_{i+1}) - o(a_{i+1}))} \times \\ &\quad \times (P_{abc, B_i})_{(k_0 - o(B_i) - o(bc) + o(a) + o(B_i) + o(bc) - o(a))}. \end{aligned}$$

The inequalities  $0 \leq k_{i+1} + o(a_{i+1}) + \dots + k_r + o(a_r) < o(A) \leq m + 1$  imply  $k_{i+1} + o(a_{i+1}) \leq m, \dots, k_r + o(a_r) \leq m$ ,  $k_0 + o(B_i) + o(bc) - o(a) \leq o(A) + o(bc) - o(a) = m + 1$  which allow to apply the inductive hypothesis to the vector fields  $\xi_{a_{i+1}}, \dots, \xi_{a_r}$  and deduce from (4.6) and (4.14) that

$$(4.15) \quad \begin{aligned} w\{[\xi_{A_i}(P_{abc, B_i})]_{(0)}\} &= k_r + o(a_r) + \dots + k_{i+1} + o(a_{i+1}) + k_0 + o(B_i) + o(bc) - o(a) \\ &= o(A) + o(bc) - o(a) = m + 1. \end{aligned}$$

Look at the homogeneous parts of order zero in (4.5) taking into account (4.15) and Definition 3.7 to get

$$(4.16) \quad w(\xi_A T_{abc}|_q) = m + 1, \quad \text{where} \quad o(A) = m + 1 - o(bc) + o(a).$$

We have consecutively:

$$\begin{aligned}
(4.17) \quad \xi_A T_{abc}|_q &= (\xi_{a_r} \dots \xi_{a_1} T_{abc})_{(0)} \\
&= \sum_{k_0+k_1+\dots+k_r=0} \xi_{a_r(k_r)} \dots \xi_{a_1(k_1)} T_{abc(k_0)} = \xi_{a_r(-o(a_r))} \dots \xi_{a_1(-o(a_1))} T_{abc(o(A))} \\
&+ \sum_{\substack{k_0+k_1+\dots+k_r=0 \\ k_0 < o(A)}} \xi_{a_r(k_r)} \dots \xi_{a_1(k_1)} T_{abc(k_0)} = X_{a_r} \dots X_{a_1} T_{abc(o(A))} + \sum_{\substack{k_0+k_1+\dots+k_r=0 \\ k_0 < o(A)}} \xi_{a_r(k_r)} \dots \xi_{a_1(k_1)} T_{abc(k_0)} \\
&= X_A T_{abc}|_q + \sum_{\substack{k_0+k_1+\dots+k_r=0 \\ k_0 < o(A)}} \xi_{a_r(k_r+o(a_r)-o(a_r))} \dots \xi_{a_1(k_1+o(a_1)-o(a_1))} T_{abc(k_0)}.
\end{aligned}$$

We are interesting in the weight of the summands at the right-hand side of (4.17). We may take  $k_0 > 0$  since the corresponding term is zero when  $k_0 = 0$ . Hence  $k_1 + o(a_1) + \dots + k_r + o(a_r) < o(A) = m+1 - o(bc) + o(a) \leq m+1$ , i.e.  $k_1 + o(a_1) \leq m, \dots, k_r + o(a_r) \leq m$  and the inductive hypothesis applied to the vector fields  $\xi_{a_i}, i = 1, \dots, r$ , gives

$$(4.18) \quad w(\xi_{a_i(k_i)}) = k_i + o(a_i), \quad i = 1, \dots, r.$$

Moreover, since  $k_0 < o(A) \leq m+1$ , the inductive hypothesis for  $T_{abc}$  yields

$$(4.19) \quad w(T_{abc(k_0)}) = k_0 + o(bc) - o(a).$$

Finally, we apply (4.16), (4.18) and (4.19) to (4.17) to get

$$(4.20) \quad w(X_A T_{abc}|_q) = m+1, \quad \text{where} \quad o(A) = m+1 - o(bc) + o(a)$$

which completes the proof that  $T_{abc(m+1-o(bc)+o(a))}$  has weight  $m+1$ .

The proof of the fact that  $R_{abcd(m+1-o(ab))}$  has weight  $m+1$  is similar and we omit it.

It follows from Proposition 3.2 and the just proved two results for the torsion and the curvature that  $\eta_{(k+2)}^i, d\eta_{(k+2)}^i, \theta_{(k+1)}^\alpha, \omega_{a(k)}^b$  have weight  $k$  for  $k = m+1$ .

Now, we check that  $\xi_{a(k-o(a))}$  and  $s_{a(k+o(b)-o(a))}^b$  have weight  $k$  for  $k = m+1$ . We have

$$\begin{aligned}
(4.21) \quad \xi_{a(m+1-o(a))} &= (s_a^b X_b)_{(m+1-o(a))} = s_{a(m+1-o(a)+o(b))}^b X_b = - \sum_{i \geq 2} s_{a(m+1+o(c)-o(a)-i)}^c \theta_{(o(b)+i)}^b (X_c) X_b,
\end{aligned}$$

where we used Lemma 4.1 and (4.2) for the second and the third equality, respectively. It is clear that

$$(4.22) \quad w(s_{a(m+1+o(c)-o(a)-i)}^c) = m+1-i,$$

because  $m+1-i \leq m-1$  and the inductive hypothesis for  $s_a^c$ . On the other hand,  $m+1+o(c)-o(a)-i \geq 0$ , which implies  $i \leq m+1$ . (The case  $i = m+2$  occurs only when  $i = m+1+o(c)-o(a)$ ,  $o(c) = 2$ ,  $o(a) = 1$ , in which situation  $s_{a(0)}^c = 0$  and the corresponding term in the sum of the right-hand side of (4.21) is zero.) Proposition 3.2 and the inductive hypothesis imply  $w(\theta_{(o(b)+i)}^b) = i$ , which, together with (4.22) and (4.21) give  $w(\xi_{a(m+1-o(a))}) = w(s_{a(m+1+o(b)-o(a))}^b) = m+1$ . This completes the induction in the case  $A = \emptyset$ .

In order to finish the proof that all the objects in (4.3) have weight  $k$ , we have to show that  $T_{abc,A(k-o(A)-o(bc)+o(a))}$  and  $R_{abcd,A(k-o(A)-o(ab))}$  have weight  $k$ . We shall do it for the torsion by a new induction on  $\#A = s$ ,  $A = (a_1, \dots, a_s)$ . For the curvature the proof is the same.

a) *Base of the induction:*  $s = 1$ . Using the results for  $\xi_{a_1}, T_{dbc}, \omega_a^d$ , it is easy to see that

$$(4.23) \quad w[(\xi_{a_1} T_{abc})_{(l)}] = w\{[\omega_a^d(\xi_{a_1}) T_{dbc}]_{(l)}\} = l + o(a_1) + o(bc) - o(a).$$

We get from (4.23) and (4.4) that  $T_{abc,a_1(l)}$  has weight  $l + o(a_1) + o(bc) - o(a)$ , which finishes the base of the induction.

b) *Inductive step.* Suppose  $w[(T_{abc,A})_{(l)}] = l + o(A) + o(bc) - o(a)$  for any multi-index  $A : \#A = s, A = (a_1, \dots, a_s)$ . Using the inductive hypothesis and the results we already have for  $\xi_a$  and  $\omega_a^b$  we obtain

$$w[(\xi_d T_{abc,A})_{(l)}] = w\{[\omega_a^e(\xi_d) T_{ebc,A}]_{(l)}\} = l + o(Ad) + o(bc) - o(a)$$



which together with the identity  $T_{abc,Ad} = \xi_d(T_{abc,A}) - P_{abc,Ad}$  show that  $(T_{abc,Ad})_{(l)}$  has weight  $l + o(Ad) + o(bc) - o(a)$ . The induction is completed.

Finally,  $S_{(m)}$  has weight  $m + 2$  since  $S = g^{\alpha\beta} g^{\gamma\delta} R_{\alpha\gamma\delta\beta}$  which ends the proof of the Lemma.  $\square$

The next result establishes a relation between the volume forms on  $(M, \eta, g, Q)$  and on  $\mathbf{G}(\mathbb{H})$ .

**Lemma 4.4.** *Let  $Vol_\eta = \frac{1}{4^n} \eta^1 \wedge \eta^2 \wedge \eta^3 \wedge (d\eta^1)^{2n}$  and  $Vol_\Theta = \frac{1}{4^n} \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge (d\Theta^1)^{2n}$  be the natural volume forms on the qc manifold  $(M, \eta, gQ)$  and the quaternionic Heisenberg group  $\mathbf{G}(\mathbb{H})$ , respectively. Then*

$$Vol_\eta = [1 + v_2 + v_3 + v_4 + O(\rho^5)] Vol_\Theta,$$

where  $v_s$  is a homogeneous polynomial of degree  $s$  and weight  $s$ ,  $s = 1, 2, 3, 4$ , and  $O(\rho^5)$  is a function in  $\mathcal{O}_{(5)}$ .

*Proof.* According to Proposition 3.2, Lemma 4.1 and Lemma 4.3, we have the following decomposition of  $\eta^i, i = 1, 2, 3$ , with respect to the base  $\{\Theta^i, dx^\alpha\}$  of  $\Lambda^1(TM)$

$$\begin{aligned} (4.24) \quad \eta^i &= \eta_{(2)}^i + \eta_{(4)}^i + \eta_{(5)}^i + \cdots = \Theta^i + \underbrace{P_{22j}^i \Theta^j + P_{32\alpha}^i dx^\alpha}_{\eta_{(4)}^i} + \underbrace{P_{33j}^i \Theta^j + P_{43\alpha}^i dx^\alpha}_{\eta_{(5)}^i} + \cdots \\ &= \Theta^i + [P_{221}^i + P_{331}^i + P_{441}^i + O(\rho^5)] \Theta^1 + [P_{222}^i + P_{332}^i + P_{442}^i + O(\rho^5)] \Theta^2 \\ &\quad + [P_{223}^i + P_{333}^i + P_{443}^i + O(\rho^5)] \Theta^3 + [P_{32\alpha}^i + P_{43\alpha}^i + O(\rho^5)] dx^\alpha, \end{aligned}$$

where  $P_{stu}^i$  is a homogeneous polynomial in  $\{x^\alpha, t^i\}$  of degree  $s$  and weight  $t$ .

Similarly, using again Proposition 3.2, Lemma 4.1 and Lemma 4.3, we get the following representation of  $d\eta^i, i = 1, 2, 3$ , with respect to the base  $\{\Theta^i \wedge \Theta^j, \Theta^i \wedge dx^\alpha, dx^\alpha \wedge dx^\beta\}$  of  $\Lambda^2(TM)$

$$\begin{aligned} (4.25) \quad d\eta^i &= d\Theta^i + \underbrace{P_{02jk}^i \Theta^j \wedge \Theta^k + P_{12j\alpha}^i \Theta^j \wedge dx^\alpha + P_{22\alpha\beta}^i dx^\alpha \wedge dx^\beta}_{d\eta_{(4)}^i} \\ &\quad + \underbrace{P_{13jk}^i \Theta^j \wedge \Theta^k + P_{23j\alpha}^i \Theta^j \wedge dx^\alpha + P_{33\alpha\beta}^i dx^\alpha \wedge dx^\beta}_{d\eta_{(5)}^i} + \cdots \\ &= d\Theta^i + [P_{02jk}^i + P_{13jk}^i + P_{24jk}^i + P_{35jk}^i + P_{46jk}^i + O(\rho^5)] \Theta^j \wedge \Theta^k + [P_{12j\alpha}^i + P_{23j\alpha}^i + P_{34j\alpha}^i + P_{45j\alpha}^i + O(\rho^5)] \Theta^j \wedge dx^\alpha \\ &\quad + [P_{22\alpha\beta}^i + P_{33\alpha\beta}^i + P_{44\alpha\beta}^i + O(\rho^5)] dx^\alpha \wedge dx^\beta, \end{aligned}$$

where  $P_{stu}^i$  is a homogeneous polynomial in  $\{x^\alpha, t^i\}$  of degree  $s$  and weight  $t$ .

It is not difficult to see, using (4.24) and (4.25), that

$$\eta^1 \wedge \eta^2 \wedge \eta^3 \wedge (d\eta^1)^{2n} = [1 + v_2 + v_3 + v_4 + O(\rho^5)] \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge (d\Theta^1)^{2n},$$

where  $v_s$  is a homogeneous polynomial of degree  $s$  and weight  $s$ ,  $s = 1, 2, 3, 4$ , which proves the lemma.  $\square$

The next essential step is to find an asymptotic expression of the qc Yamabe functional (1.1) over a special set of "test functions".

It is shown in [15, 17, 18] that the function  $F := [(1 + |p|^2)^2 + |w|^2]^{-(n+1)}$  is an extremal for the qc Yamabe functional  $\Upsilon_\Theta$  on the quaternionic Heisenberg group  $\mathbf{G}(\mathbb{H})$ . Here  $|p|^2 := (x^1)^2 + \cdots + (x^{4n})^2$ ,  $|w|^2 := (t^1)^2 + (t^2)^2 + (t^3)^2$ . The function  $F^{\varepsilon, \lambda} := \varepsilon^\lambda \delta_{1/\varepsilon}^* F$ ,  $\varepsilon, \lambda \in \mathbb{R}$ ,  $\varepsilon > 0$ , is also an extremal. Let us choose  $\lambda = -2(n+1)$  and denote  $F^\varepsilon := F^{\varepsilon, -2(n+1)} = \varepsilon^{2(n+1)} [(\varepsilon^2 + |p|^2)^2 + |w|^2]^{-(n+1)}$ . It is not difficult to see that the integral  $\int_{\mathbf{G}(\mathbb{H})} (F^\varepsilon)^{2^*} Vol_\Theta$  is a constant independent of  $\varepsilon$ . Indeed, we have consecutively:

$$\begin{aligned} \int_{\mathbf{G}(\mathbb{H})} (F^\varepsilon)^{2^*} Vol_\Theta &= \int_{\mathbf{G}(\mathbb{H})} \varepsilon^{-4n-6} (\delta_{1/\varepsilon}^* F)^{2^*} Vol_\Theta = \int_{\mathbf{G}(\mathbb{H})} (\delta_{1/\varepsilon}^* F)^{2^*} \delta_{1/\varepsilon}^* (Vol_\Theta) \\ &= \int_{\mathbf{G}(\mathbb{H})} \delta_{1/\varepsilon}^* (F^{2^*} Vol_\Theta) = \int_{\mathbf{G}(\mathbb{H})} F^{2^*} Vol_\Theta. \end{aligned}$$

The natural distance function on  $\mathbf{G}(\mathbb{H})$  is defined in the coordinates  $\{x^\alpha, t^i\}$  by  $\rho := \sqrt[4]{|p|^4 + |t|^2} = \sqrt[4]{[(x^1)^2 + \dots + (x^{4n})^2]^2 + (t^1)^2 + (t^2)^2 + (t^3)^2}$ . The polar change of the coordinates  $\{x^\alpha, t^i\}$  is defined by

$$(4.26) \quad \begin{aligned} x^1 &= \rho \xi^1, \dots, x^{4n} = \rho \xi^{4n}, t^1 = \rho^2 \tau^1, t^2 = \rho^2 \tau^2, t^3 = \rho^2 \tau^3, \quad \text{where} \\ &[(\xi^1)^2 + \dots + (\xi^{4n})^2]^2 + (\tau^1)^2 + (\tau^2)^2 + (\tau^3)^2 = 1. \end{aligned}$$

Let  $\{x^\alpha, t^i\}$  be qc parabolic normal coordinates near  $q \in M$  for a contact form  $\eta$ . We may assume the definition area to be  $\{\rho < 2k\}$  for a suitable  $k > 0$ . Following [22], we define the function  $\psi \in Co^\infty(M)$  with compact support in the set  $\{\rho < 2k\}$  and  $\psi \equiv 1$  in the set  $\{\rho < k\}$ . The "test function"  $f^\varepsilon$  is defined by  $f^\varepsilon := \psi F^\varepsilon, \varepsilon > 0$ , on the set  $\{\rho < 2k\}$ . The main result of this section is contained in the following

**Proposition 4.5.** *For the qc Yamabe functional (1.1) over the test functions  $f^\varepsilon$  the next asymptotic expression holds*

$$(4.27) \quad \Upsilon_\eta(f^\varepsilon) = \frac{b_0(n) + b_4(n) \|W^{qc}\|^2 \varepsilon^4 + O(\varepsilon^5)}{(a_0(n) + a_4(n) \|W^{qc}\|^2 \varepsilon^4 + O(\varepsilon^5))^{2/2^*}},$$

where  $a_0(n), a_4(n), b_0(n)$  and  $b_4(n)$  are some independent of the qc structure dimensional constants.

*Proof.* We shall examine separately each of the integrals that appear in the expression

$$(4.28) \quad \Upsilon_\eta(f^\varepsilon) = \frac{\int_M [4 \frac{n+2}{n+1} |\nabla f^\varepsilon|_\eta^2 + S(f^\varepsilon)^2] Vol_\eta}{[\int_M (f^\varepsilon)^{2^*} Vol_\eta]^{2/2^*}}.$$

A) We begin with  $\int_M (f^\varepsilon)^{2^*} Vol_\eta$ .

If  $\varphi$  is an integrable function on  $\mathbf{G}(\mathbb{H})$  and  $|\varphi| \leq C\Phi(\rho)$  for a constant  $C$  and an integrable function  $\Phi$  on  $\mathbf{G}(\mathbb{H})$  that depends only on  $\rho$  then the next formula holds

$$(4.29) \quad \int_{a < \rho < b} \varphi Vol_\Theta = O\left(\int_a^b \Phi(\rho) \rho^{4n+5} d\rho\right).$$

Indeed, the polar change (4.26) of the coordinates in the volume form on  $\mathbf{G}(\mathbb{H})$  yields  $Vol_\Theta = \rho^{4n+5} d\rho \wedge d\sigma$ , where  $d\sigma$  is a  $4n+2$ -form that depends only on  $\xi^1, \dots, \xi^{4n}, \tau^1, \tau^2, \tau^3$  and (4.29) follows. We also have

$$\begin{aligned} F^{-\frac{1}{n+1}} &= (1 + |p|^2)^2 + |w|^2 = [1 + (x^1)^2 + \dots + (x^{4n})^2]^2 + (t^1)^2 + (t^2)^2 + (t^3)^2 \\ &\geq 1 + \rho^4 [(\xi^1)^2 + \dots + (\xi^{4n})^2]^2 + \rho^4 [(\tau^1)^2 + (\tau^2)^2 + (\tau^3)^2] = 1 + \rho^4 \geq \tilde{C}(1 + \rho)^4 \end{aligned}$$

for some positive constant  $\tilde{C}$  which implies

$$(4.30) \quad |F| \leq C(1 + \rho)^{-4(n+1)}.$$

We get consecutively:

$$\begin{aligned}
(4.31) \quad \int_M (f^\varepsilon)^{2^*} Vol_\eta &= \int_{\rho < k} (F^\varepsilon)^{2^*} [1 + v_2 + v_3 + v_4 + O(\rho^5)] Vol_\Theta \\
&\quad + \int_{k < \rho < 2k} \psi^{2^*} (F^\varepsilon)^{2^*} [1 + v_2 + v_3 + v_4 + O(\rho^5)] Vol_\Theta \\
&= \int_{\rho < k/\varepsilon} (\delta_\varepsilon^* F^\varepsilon)^{2^*} [1 + \varepsilon^2 v_2 + \varepsilon^3 v_3 + \varepsilon^4 v_4 + O(\varepsilon^5 \rho^5)] \delta_\varepsilon^* Vol_\Theta \\
&\quad + \int_{k/\varepsilon < \rho < 2k/\varepsilon} \delta_\varepsilon^* \psi^{2^*} (\delta_\varepsilon^* F^\varepsilon)^{2^*} [1 + \varepsilon^2 v_2 + \varepsilon^3 v_3 + \varepsilon^4 v_4 + O(\varepsilon^5 \rho^5)] \delta_\varepsilon^* Vol_\Theta \\
&= \int_{\rho < k/\varepsilon} F^{2^*} [1 + \varepsilon^2 v_2 + \varepsilon^3 v_3 + \varepsilon^4 v_4 + O(\varepsilon^5 \rho^5)] Vol_\Theta \\
&\quad + \int_{k/\varepsilon < \rho < 2k/\varepsilon} \delta_\varepsilon^* \psi^{2^*} F^{2^*} [1 + \varepsilon^2 v_2 + \varepsilon^3 v_3 + \varepsilon^4 v_4 + O(\varepsilon^5 \rho^5)] Vol_\Theta \\
&= \int_{\rho < k/\varepsilon} F^{2^*} [1 + \varepsilon^2 v_2 + \varepsilon^3 v_3 + \varepsilon^4 v_4 + O(\varepsilon^5 \rho^5)] Vol_\Theta + O\left(\int_{k/\varepsilon < \rho < 2k/\varepsilon} F^{2^*} Vol_\Theta\right) \\
&= \int_{\mathbf{G}(\mathbb{H})} F^{2^*} (1 + \varepsilon^2 v_2 + \varepsilon^3 v_3 + \varepsilon^4 v_4) Vol_\Theta - \int_{\rho > k/\varepsilon} F^{2^*} (1 + \varepsilon^2 v_2 + \varepsilon^3 v_3 + \varepsilon^4 v_4) Vol_\Theta + \int_{\rho < k/\varepsilon} F^{2^*} O(\varepsilon^5 \rho^5) Vol_\Theta \\
&\quad + O\left(\int_{k/\varepsilon < \rho < 2k/\varepsilon} F^{2^*} Vol_\Theta\right) = \int_{\mathbf{G}(\mathbb{H})} F^{2^*} (1 + \varepsilon^2 v_2 + \varepsilon^3 v_3 + \varepsilon^4 v_4) Vol_\Theta \\
&\quad + O\left(\int_{k/\varepsilon}^\infty \sum_{i=0}^4 \varepsilon^i \rho^i (1 + \rho)^{-4(2n+3)} \rho^{4n+5} d\rho\right) + O\left(\int_0^{k/\varepsilon} \varepsilon^5 \rho^5 (1 + \rho)^{-4(2n+3)} \rho^{4n+5} d\rho\right) \\
&\quad + O\left(\int_{k/\varepsilon}^{2k/\varepsilon} (1 + \rho)^{-4(2n+3)} \rho^{4n+5} d\rho\right),
\end{aligned}$$

where we used the definition of  $f^\varepsilon$  and Lemma 4.4 for the first equality and the parabolic dilation change of the variables for the second one. The third identity follows from the definition of  $F^\varepsilon$ . To get the fourth one, we used the next chain of relations:

$$\begin{aligned}
&\left| \int_{k/\varepsilon < \rho < 2k/\varepsilon} \delta_\varepsilon^* \psi^{2^*} F^{2^*} [1 + \varepsilon^2 v_2 + \varepsilon^3 v_3 + \varepsilon^4 v_4 + O(\varepsilon^5 \rho^5)] Vol_\Theta \right| \\
&\leq \int_{k/\varepsilon < \rho < 2k/\varepsilon} C_0 F^{2^*} |1 + \varepsilon^2 v_2 + \varepsilon^3 v_3 + \varepsilon^4 v_4 + O(\varepsilon^5 \rho^5)| Vol_\Theta \\
&\leq \int_{k/\varepsilon < \rho < 2k/\varepsilon} C_0 F^{2^*} [1 + \varepsilon^2 \rho^2 + \varepsilon^3 \rho^3 + \varepsilon^4 \rho^4 + O(\varepsilon^5 \rho^5)] Vol_\Theta \\
&\leq \int_{k/\varepsilon < \rho < 2k/\varepsilon} C_0 F^{2^*} \underbrace{[1 + (2k)^2 + (2k)^3 + (2k)^4 + O((2k)^5)]}_{\text{convergent series}} Vol_\Theta = C_1 \int_{k/\varepsilon < \rho < 2k/\varepsilon} F^{2^*} Vol_\Theta,
\end{aligned}$$

where  $C_0$  and  $C_1$  are some suitable positive constants and  $k$  is chosen sufficient small. Hence,

$$\int_{k/\varepsilon < \rho < 2k/\varepsilon} \delta_\varepsilon^* \psi^{2^*} F^{2^*} [1 + \varepsilon^2 v_2 + \varepsilon^3 v_3 + \varepsilon^4 v_4 + O(\varepsilon^5 \rho^5)] Vol_\Theta = O\left(\int_{k/\varepsilon < \rho < 2k/\varepsilon} F^{2^*} Vol_\Theta\right).$$

The fifth identity in (4.31) is clear while the sixth one is obtained from (4.29) and (4.30).

For the first term at the right-hand side of (4.31) we have

$$\int_{\mathbf{G}(\mathbb{H})} F^{2^*} \underbrace{\left(1 + \varepsilon^2 v_2 + \varepsilon^3 v_3 + \varepsilon^4 v_4\right)}_{=: v_0} Vol_\Theta = c_0 + c_2 \varepsilon^2 + c_3 \varepsilon^3 + c_4 \varepsilon^4,$$

where  $c_i := \int_{\mathbf{G}(\mathbb{H})} F^{2*} v_i \text{Vol}_\Theta$ ,  $i = 0, 2, 3, 4$ , is a quaternionic contact invariant scalar quantity of weight  $i$ . It follows by Theorem 3.8 that  $c_2 = c_3 = 0$  and therefore

$$(4.32) \quad \int_{\mathbf{G}(\mathbb{H})} F^{2*} (1 + \varepsilon^2 v_2 + \varepsilon^3 v_3 + \varepsilon^4 v_4) \text{Vol}_\Theta = a_0(n) + a_4(n) \|W^{qc}\|^2 \varepsilon^4,$$

where  $a_0(n)$  and  $a_4(n)$  are some dimensional constants, independent on the qc structure.

Finally, we are interesting in the last three expressions at the right-hand side of (4.31). We have for  $i = 0, \dots, 4$  that  $\rho^{4n+5+i}(1+\rho)^{-8n-12} \leq \rho^{-6}$  and consequently,

$$(4.33) \quad \int_{k/\varepsilon}^{\infty} \sum_{i=0}^4 \varepsilon^i \rho^i (1+\rho)^{-4(2n+3)} \rho^{4n+5} d\rho = O(\varepsilon^5).$$

In a similar way, we obtain

$$(4.34) \quad \int_0^{k/\varepsilon} \varepsilon^5 \rho^5 (1+\rho)^{-4(2n+3)} \rho^{4n+5} d\rho = O(\varepsilon^5), \quad \int_{k/\varepsilon}^{2k/\varepsilon} (1+\rho)^{-4(2n+3)} \rho^{4n+5} d\rho = O(\varepsilon^5).$$

A substitution of (4.32), (4.33) and (4.34) into (4.31) gives

$$(4.35) \quad \int_M (f^\varepsilon)^{2*} \text{Vol}_\eta = a_0(n) + a_4(n) \|W^{qc}\|^2 \varepsilon^4 + O(\varepsilon^5).$$

B) We continue with  $\int_M |\nabla f^\varepsilon|_\eta^2 \text{Vol}_\eta$ . It follows directly by the decomposition of  $\xi_a$  determined in Lemma 4.2, the definition of the function  $f^\varepsilon$  and Lemma 4.4 that

$$(4.36) \quad \begin{aligned} & \int_M |\nabla f^\varepsilon|_\eta^2 \text{Vol}_\eta \\ &= \int_M \sum_{\alpha=1}^{4n} (\xi_\alpha f^\varepsilon)^2 \text{Vol}_\eta = \int_{\rho < k} \sum_{\alpha=1}^{4n} s_\alpha^a s_\alpha^b (X_a F^\varepsilon)(X_b F^\varepsilon) [1 + v_2 + v_3 + v_4 + O(\rho^5)] \text{Vol}_\Theta \\ &+ \int_{k < \rho < 2k} \sum_{\alpha=1}^{4n} s_\alpha^a s_\alpha^b [(X_a \psi) F^\varepsilon + \psi X_a (F^\varepsilon)] [(X_b \psi) F^\varepsilon + \psi X_b (F^\varepsilon)] [1 + v_2 + v_3 + v_4 + O(\rho^5)] \text{Vol}_\Theta =: I_1 + I_2. \end{aligned}$$

We begin with  $I_1$ . At first, note that the following representation holds

$$\sum_{\alpha=1}^{4n} s_\alpha^a s_\alpha^b = \sum_{\alpha=1}^{4n} \sum_{m=0}^{\infty} \sum_{k_1+k_2=m} s_{\alpha(k_1+o(a)-1)}^a s_{\alpha(k_2+o(b)-1)}^b =: \sum_{m=0}^{\infty} v_m^{ab},$$

where  $v_m^{ab}$ ,  $m = 0, 1, \dots, \infty$ , is a homogeneous polynomial of degree  $m + o(ab) - 2$  and weight  $m$ . We have

$$(4.37) \quad \begin{aligned} I_1 &= \int_{\rho < k} \left( \sum_{m=0}^{\infty} v_m^{ab} \right) (X_a F^\varepsilon)(X_b F^\varepsilon) [1 + v_2 + v_3 + v_4 + O(\rho^5)] \text{Vol}_\Theta \\ &= \int_{\rho < k} [w_0^{ab} + w_1^{ab} + w_2^{ab} + w_3^{ab} + w_4^{ab} + O(\rho^{3+o(ab)})] (X_a F^\varepsilon)(X_b F^\varepsilon) \text{Vol}_\Theta, \end{aligned}$$

where the homogeneous polynomial  $w_i^{ab}$ ,  $i = 0, \dots, 4$ , is formed from the polynomials  $v_m^{ab}$ ,  $v_j$  and is of degree  $i + o(ab) - 2$  and of weight  $i$ . Moreover, it follows directly by the definition of  $F^\varepsilon$  that

$$(4.38) \quad \delta_\varepsilon^*(X_a F^\varepsilon) = \varepsilon^{-2(n+1)-o(a)} X_a F.$$

Another fact we shall need is the inequality

$$(4.39) \quad |X_a F| \leq C(1+\rho)^{-4(n+1)-o(a)},$$

where  $C$  is some positive constant.<sup>1</sup> In order to check (4.39), we suppose firstly that  $a \in \{1, \dots, 4n\}$ . Using the definitions of the function  $F$  and the vector field  $X_a$ , we obtain

$$X_a F = -4(n+1)\{[1 + (x^1)^2 + \dots + (x^{4n})^2]^2 + (t^1)^2 + (t^2)^2 + (t^3)^2\}^{-n-2} \times \\ \times \{[1 + (x^1)^2 + \dots + (x^{4n})^2]x^a + I_{\beta a}^1 x^\beta t^1 + I_{\beta a}^2 x^\beta t^2 + I_{\beta a}^3 x^\beta t^3\}.$$

The polar change (4.26) in the above identity gives  $|X_a F| \leq \tilde{C}(1+\rho)^{-4(n+2)}(\rho + \rho^3)$  for some positive constant  $\tilde{C}$ , yielding  $|X_a F| \leq C(1+\rho)^{-4n-5} = C(1+\rho)^{-4(n+1)-o(a)}$ , where  $C$  is a positive constant. Thus (4.39) is proved for  $a \in \{1, \dots, 4n\}$ . The case  $a \in \{4n+1, 4n+2, 4n+3\}$  is considered in a similar way.

We return to (4.37), make the parabolic dilations change  $\delta_\varepsilon(x^\alpha, t^i) = (\varepsilon x^\alpha, \varepsilon^2 t^i)$  and use (4.38) to get

$$(4.40) \quad I_1 = \int_{\rho < k/\varepsilon} \sum_{i=0}^4 \varepsilon^i w_i^{ab}(X_a F)(X_b F) Vol_\Theta + \int_{\rho < k/\varepsilon} O(\varepsilon^5 \rho^{3+o(ab)})(X_a F)(X_b F) Vol_\Theta \\ = \int_{\mathbf{G}(\mathbb{H})} \sum_{i=0}^4 \varepsilon^i w_i^{ab}(X_a F)(X_b F) Vol_\Theta - \int_{\rho > k/\varepsilon} \sum_{i=0}^4 \varepsilon^i w_i^{ab}(X_a F)(X_b F) Vol_\Theta \\ + \int_{\rho < k/\varepsilon} O(\varepsilon^5 \rho^{3+o(ab)})(X_a F)(X_b F) Vol_\Theta.$$

Now we shall examine the three integrals in the right-hand side of (4.40). We have

$$\int_{\mathbf{G}(\mathbb{H})} \sum_{i=0}^4 \varepsilon^i w_i^{ab}(X_a F)(X_b F) Vol_\Theta = d_0 + \varepsilon d_1 + \varepsilon^2 d_2 + \varepsilon^3 d_3 + \varepsilon^4 d_4,$$

where  $d_i := \int_{\mathbf{G}(\mathbb{H})} w_i^{ab}(X_a F)(X_b F) Vol_\Theta$ ,  $i = 0, \dots, 4$ , is a quaternionic contact invariant scalar quantity of weight  $i$ . In the same manner as in (4.32) we obtain by Theorem 3.8 that

$$(4.41) \quad \int_{\mathbf{G}(\mathbb{H})} \sum_{i=0}^4 \varepsilon^i w_i^{ab}(X_a F)(X_b F) Vol_\Theta = \tilde{b}_0(n) + \tilde{b}_4(n) \|W^{qc}\|^2 \varepsilon^4,$$

where  $\tilde{b}_0(n)$  and  $\tilde{b}_4(n)$  are some dimensional constants, independent of the qc structure.

We get using (4.29) and (4.39) that

$$| \int_{\rho > k/\varepsilon} \sum_{i=0}^4 \varepsilon^i w_i^{ab}(X_a F)(X_b F) Vol_\Theta | \leq C \int_{\rho > k/\varepsilon} \sum_{i=0}^4 \varepsilon^i \rho^{i+o(ab)-2} (1+\rho)^{-8(n+1)-o(ab)} Vol_\Theta \\ = O\left[\sum_{i=0}^4 \varepsilon^i \int_{k/\varepsilon}^\infty (\rho^{4n+3+i+o(ab)} (1+\rho)^{-8(n+1)-o(ab)}) d\rho\right]$$

which combined with the inequality  $\rho^{4n+3+i+o(ab)} (1+\rho)^{-8(n+1)-o(ab)} \leq \rho^{-6+i}$  lead to

$$(4.42) \quad \int_{\rho > k/\varepsilon} \sum_{i=0}^4 \varepsilon^i w_i^{ab}(X_a F)(X_b F) Vol_\Theta = O(\varepsilon^5).$$

To handle the last integral in the right-hand side of (4.40), we use (4.39) to get

$$| \int_{\rho < k/\varepsilon} O(\varepsilon^5 \rho^{3+o(ab)})(X_a F)(X_b F) Vol_\Theta | \\ \leq C \int_{\rho < k/\varepsilon} \varepsilon^5 (\rho^{3+o(ab)} + \varepsilon \rho^{4+o(ab)} + \dots) (1+\rho)^{-8(n+1)-o(ab)} Vol_\Theta,$$

which together with  $\rho^{3+o(ab)} + \varepsilon \rho^{4+o(ab)} + \dots \leq \tilde{C} \rho^{3+o(ab)}$  for  $\rho < \frac{k}{\varepsilon}$ ,  $k$  sufficiently small, and (4.29) give

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<sup>1</sup>We shall use again the letters  $C$  and  $\tilde{C}$  to denote some (positive) constants. We are doing this now and later in order to avoid the excessive accumulation of different letters and indices.

$$\int_{\rho < k/\varepsilon} O(\varepsilon^5 \rho^{3+o(ab)})(X_a F)(X_b F) Vol_\Theta = O[\varepsilon^5 \int_0^{k/\varepsilon} \rho^{4n+8+o(ab)}(1+\rho)^{-8(n+1)-o(ab)} d\rho].$$

Now, it is easy to see after some standard analysis that

$$(4.43) \quad \int_{\rho < k/\varepsilon} O(\varepsilon^5 \rho^{3+o(ab)})(X_a F)(X_b F) Vol_\Theta = O(\varepsilon^5).$$

We substitute (4.41), (4.42) and (4.43) into (4.40) to obtain

$$(4.44) \quad I_1 = \tilde{b}_0(n) + \tilde{b}_4(n) \|W^{qc}\|^2 \varepsilon^4 + O(\varepsilon^5).$$

To manage the integral  $I_2$  in (4.36), we note that for  $k < \rho < 2k$  the following inequalities hold  $|s_\alpha^a| |s_\alpha^b| \leq M_\alpha^{ab}$  and  $|1 + v_2 + v_3 + v_4 + O(\rho^5)| \leq N$  for suitable constants  $M_\alpha^{ab}, N$ . Then we have

$$|I_2| \leq \int_{k < \rho < 2k} \sum_{\alpha=1}^{4n} C_\alpha^{ab} (|F^\varepsilon|^2 + |X_a F^\varepsilon| |F^\varepsilon| + |X_b F^\varepsilon| |F^\varepsilon| + |X_a F^\varepsilon| |X_b F^\varepsilon|) Vol_\Theta$$

for some positive constants  $C_\alpha^{ab}$ . The latter yields

$$(4.45) \quad I_2 = O\left(\int_{k < \rho < 2k} |F^\varepsilon|^2 Vol_\Theta\right) + O\left(\sum_{a=1}^{4n+3} \int_{k < \rho < 2k} |X_a F^\varepsilon| |F^\varepsilon| Vol_\Theta\right) + O\left(\sum_{a,b=1}^{4n+3} \int_{k < \rho < 2k} |X_a F^\varepsilon| |X_b F^\varepsilon| Vol_\Theta\right).$$

We examine the three integrals in the right-hand side of (4.45). We begin with

$$\int_{k < \rho < 2k} |F^\varepsilon|^2 Vol_\Theta = \varepsilon^2 \int_{k/\varepsilon < \rho < 2k/\varepsilon} |F|^2 Vol_\Theta = \varepsilon^2 O\left(\int_{k/\varepsilon}^{2k/\varepsilon} (1+\rho)^{-8(n+1)} \rho^{4n+5} d\rho\right),$$

where we applied the parabolic dilation change of the variables to obtain the first equality, (4.29) and (4.30) to get the second one. The latter together with the inequality  $(1+\rho)^{-8(n+1)} \rho^{4n+5} \leq \rho^{-4}$  lead to

$$(4.46) \quad \int_{k < \rho < 2k} |F^\varepsilon|^2 Vol_\Theta = O(\varepsilon^5).$$

Similarly, we obtain using (4.29), (4.30), (4.38) and (4.39) the next two relations

$$(4.47) \quad \int_{k < \rho < 2k} |X_a F^\varepsilon| |F^\varepsilon| Vol_\Theta = O(\varepsilon^5), \quad \int_{k < \rho < 2k} |X_a F^\varepsilon| |X_b F^\varepsilon| Vol_\Theta = O(\varepsilon^5).$$

A substitution of (4.46), (4.47) into (4.45) gives

$$(4.48) \quad I_2 = O(\varepsilon^5)$$

which combined with (4.44) imply

$$(4.49) \quad \int_M |\nabla f^\varepsilon|_\eta^2 Vol_\eta = \tilde{b}_0(n) + \tilde{b}_4(n) \|W^{qc}\|^2 \varepsilon^4 + O(\varepsilon^5).$$

C) Finally, it remains to investigate the integral  $\int_M S(f^\varepsilon)^2 Vol_\eta$  that appears in (4.28). We have

$$(4.50) \quad \begin{aligned} \int_M S(f^\varepsilon)^2 Vol_\eta &= \int_{\mathbf{G}(\mathbb{H})} [S_{(0)} + S_{(1)} + S_{(2)} + O(\rho^3)] (f^\varepsilon)^2 [1 + v_2 + O(\rho^3)] Vol_\Theta \\ &= \int_{\rho < k} [S_{(0)} + S_{(1)} + S_{(2)} + O(\rho^3)] (F^\varepsilon)^2 [1 + v_2 + O(\rho^3)] Vol_\Theta \\ &\quad + \int_{k < \rho < 2k} [S_{(0)} + S_{(1)} + S_{(2)} + O(\rho^3)] \psi^2 (F^\varepsilon)^2 [1 + v_2 + O(\rho^3)] Vol_\Theta =: J_1 + J_2, \end{aligned}$$

where we used Lemma 4.4 to get the first equality and the definition of  $f^\varepsilon$  to obtain the second one.



We examine separately the integrals  $J_1$  and  $J_2$  in (4.50). For the integral  $J_1$  we have

$$(4.51) \quad J_1 = \int_{\rho < k/\varepsilon} [S_{(0)}\varepsilon^2 + S_{(1)}\varepsilon^3 + (S_{(0)}v_2 + S_{(2)})\varepsilon^4 + O(\rho^3\varepsilon^5)] F^2 Vol_\Theta \\ = \int_{\mathbf{G}(\mathbb{H})} [S_{(0)}\varepsilon^2 + S_{(1)}\varepsilon^3 + (S_{(0)}v_2 + S_{(2)})\varepsilon^4] F^2 Vol_\Theta - \int_{\rho > k/\varepsilon} [S_{(0)}\varepsilon^2 + S_{(1)}\varepsilon^3 + (S_{(0)}v_2 + S_{(2)})\varepsilon^4] F^2 Vol_\Theta \\ + \int_{\rho < k/\varepsilon} O(\rho^3\varepsilon^5) F^2 Vol_\Theta.$$

We handle the three integrals that stay in the right-hand side of (4.51). We have firstly

$$\int_{\mathbf{G}(\mathbb{H})} [S_{(0)}\varepsilon^2 + S_{(1)}\varepsilon^3 + (S_{(0)}v_2 + S_{(2)})\varepsilon^4] F^2 Vol_\Theta = \varepsilon^2 e_2 + \varepsilon^3 e_3 + \varepsilon^4 e_4,$$

where  $e_2 := \int_{\mathbf{G}(\mathbb{H})} S_{(0)} F^2 Vol_\Theta$ ,  $e_3 := \int_{\mathbf{G}(\mathbb{H})} S_{(1)} F^2 Vol_\Theta$  and  $e_4 := \int_{\mathbf{G}(\mathbb{H})} (S_{(0)}v_2 + S_{(2)}) F^2 Vol_\Theta$  are some quaternionic contact invariant scalar quantities of weight 2, 3 and 4, respectively. It follows from Theorem 3.8 that  $e_2 = e_3 = 0$ . To show  $e_4 = 0$  we use that  $S_{(0)} = S|_q = 0$  by Theorem 3.6. Furthermore,

$$S_{(2)} = \sum_{o(A)=2} \frac{1}{(\#A)!} \left(\frac{1}{2}\right)^{o(A)-\#A} x^A (X_A S)|_q = \sum_{o(A)=2} \frac{1}{(\#A)!} \left(\frac{1}{2}\right)^{o(A)-\#A} x^A (S_{,A})|_q = \sum_{\alpha,\beta} \frac{1}{2} x^\alpha x^\beta (S_{,\alpha\beta})|_q.$$

Indeed, Proposition 3.4 implies the first equality. The second one follows from the fact that  $S \in \mathcal{O}_1$  and Lemma 3.5. The third identity is a consequence of the equality  $S_{,\tilde{\alpha}}|_q = 0$ ,  $\tilde{\alpha} \in \{4n+1, 4n+2, 4n+3\}$ , see Theorem 3.6. So, we obtain  $e_4 = S_{,\alpha\beta}|_q c^{\alpha\beta}$ , where  $c^{\alpha\beta} := \frac{1}{2} \int_{\mathbf{G}(\mathbb{H})} x^\alpha x^\beta F^2 Vol_\Theta$  is a quaternionic contact scalar invariant quantity of weight 0. But the only qc scalar invariant quantities that can be formed by a complete contraction of  $S_{,\alpha\beta}$  are  $S_{,\alpha}{}^\alpha$  and  $S_{,\tilde{\alpha}}{}^{\tilde{\alpha}}$  (see [23], p. 30), which are zero at  $q$  because of Theorem 3.6. Thus  $e_4 = 0$  and we conclude that

$$(4.52) \quad \int_{\mathbf{G}(\mathbb{H})} [S_{(0)}\varepsilon^2 + S_{(1)}\varepsilon^3 + (S_{(0)}v_2 + S_{(2)})\varepsilon^4] F^2 Vol_\Theta = 0.$$

To deal with the second integral in the right-hand side of (4.51), we are based on the equality

$$\int_{\rho > k/\varepsilon} [S_{(0)}\varepsilon^2 + S_{(1)}\varepsilon^3 + (S_{(0)}v_2 + S_{(2)})\varepsilon^4] F^2 Vol_\Theta = O\left[\int_{k/\varepsilon}^\infty \left(\sum_{i=0}^2 \rho^i \varepsilon^{i+2}\right) (1+\rho)^{-8(n+1)} \rho^{4n+5} d\rho\right],$$

following from (4.29) and (4.30). This limiting behavior together with  $(1+\rho)^{-8(n+1)} \rho^{4n+5+i} \leq \rho^{i-4}$  yield

$$(4.53) \quad \int_{\rho > k/\varepsilon} [S_{(0)}\varepsilon^2 + S_{(1)}\varepsilon^3 + (S_{(0)}v_2 + S_{(2)})\varepsilon^4] F^2 Vol_\Theta = O(\varepsilon^5).$$

To handle the third integral in the right-hand side of (4.51), we observe at first the equality

$$\int_{\rho < k/\varepsilon} O(\rho^3 \varepsilon^5) F^2 Vol_\Theta = O\left[\varepsilon^5 \int_0^\infty \rho^{4n+8} (1+\rho)^{-8(n+1)} d\rho\right],$$

which is a direct consequence from (4.29) and (4.30). It is not difficult to see that this identity together with the inequality  $\rho^{4n+8} (1+\rho)^{-8(n+1)}$  imply

$$(4.54) \quad \int_{\rho < k/\varepsilon} O(\rho^3 \varepsilon^5) F^2 Vol_\Theta = O(\varepsilon^5).$$

We substitute (4.52), (4.53) and (4.54) in (4.51) to obtain

$$(4.55) \quad J_1 = O(\varepsilon^5).$$

It remains to investigate the integral  $J_2$  from (4.50). We apply the parabolic dilations change of the variables to  $J_2$  and use (4.29) and (4.30) to get after some standard calculations

$$J_2 = O\left[\int_{k/\varepsilon}^{2k/\varepsilon} \underbrace{(1 + \varepsilon\rho + \varepsilon^2\rho^2 + O(\varepsilon^3\rho^3))}_{\leq Const} \varepsilon^2 (1+\rho)^{-8(n+1)} \rho^{4n+5} d\rho\right] = O\left[\varepsilon^2 \int_{k/\varepsilon}^{2k/\varepsilon} (1+\rho)^{-8(n+1)} \rho^{4n+4} d\rho\right].$$

The latter combined with the inequality  $(1 + \rho)^{-8(n+1)} \rho^{4n+5} < \rho^{-4}$  yields  $J_2 = O(\varepsilon^5)$  which combined with (4.50) and (4.55) implies

$$(4.56) \quad \int_M S(f^\varepsilon)^2 \text{Vol}_\eta = O(\varepsilon^5).$$

We substitute (4.35), (4.49) and (4.56) in (4.28) and set  $b_0(n) := 4 \frac{n+2}{n+1} \tilde{b}_0(n)$  and  $b_4(n) := 4 \frac{n+2}{n+1} \tilde{b}_4(n)$ , which ends the proof of Proposition 4.5.  $\square$

## 5. EXPLICIT EVALUATION OF CONSTANTS

The aim of our investigations in this section is to calculate explicitly the constants  $a_0(n), a_4(n), b_0(n)$  and  $b_4(n)$  that stay at the right-hand side of (4.27). We begin with an algebraic lemma.

**Lemma 5.1.** *If  $\omega = \omega_{\alpha\beta} dx^\alpha \wedge dx^\beta$  is a 2-form, then<sup>2</sup>:*

- a)  $2n\Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge \omega \wedge (d\Theta^1)^{2n-1} = \text{tr}\omega \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge (d\Theta^1)^{2n}$ , where  $\text{tr}\omega := \omega_{12} + \dots + \omega_{4n-1,4n} = -\frac{1}{2}I_{\alpha\beta}^1 \omega_{\alpha\beta}$ ;
- b)  $8n(2n-1)\Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge \omega^2 \wedge (d\Theta^1)^{2n-2} = (I_{\alpha\beta}^1 I_{\gamma\delta}^1 + 2I_{\alpha\delta}^1 I_{\beta\gamma}^1) \omega_{\alpha\beta} \omega_{\gamma\delta} \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge (d\Theta^1)^{2n}$ .

*Proof.* The claim a) follows from the next equalities

$$\begin{aligned} \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge \omega \wedge (d\Theta^1)^{2n-1} &= \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge (\omega_{\alpha\beta} dx^\alpha \wedge dx^\beta) \wedge (-I_{\alpha\beta}^1 dx^\alpha \wedge dx^\beta)^{2n-1} \\ &= \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge (2\omega_{12} dx^1 \wedge dx^2 + \dots + 2\omega_{4n-1,4n} dx^{4n-1} \wedge dx^{4n}) \wedge (2dx^1 \wedge dx^2 + \dots + 2dx^{4n-1} \wedge dx^{4n})^{2n-1} \\ &= \frac{1}{2n} \text{tr}\omega \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge (d\Theta^1)^{2n}. \end{aligned}$$

To handle b), we note that the following formula holds:

$$(5.1) \quad 8n(2n-1)\Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge dx^l \wedge dx^{l+1} \wedge dx^m \wedge dx^{m+1} \wedge (d\Theta^1)^{2n-2} = \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge (d\Theta^1)^{2n},$$

where  $l, m \in \{1, 3, \dots, 4n-1\}, l \neq m$ . We obtain consecutively

$$\begin{aligned} \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge \omega^2 \wedge (d\Theta^1)^{2n-2} &= \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge (\omega_{\alpha\beta} \omega_{\gamma\delta} dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\delta) \wedge (d\Theta^1)^{2n-2} \\ &= \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge [8(\omega_{12}\omega_{34} - \omega_{13}\omega_{24} + \omega_{14}\omega_{23})dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \\ &\quad + \dots + 8(\omega_{4n-3,4n-2}\omega_{4n-1,4n} - \omega_{4n-3,4n-1}\omega_{4n-2,4n} + \omega_{4n-3,4n}\omega_{4n-2,4n-1})dx^{4n-3} \wedge dx^{4n-2} \wedge dx^{4n-1} \wedge dx^{4n}] \wedge \\ &\quad \wedge (d\Theta^1)^{2n-2} = \frac{1}{n(2n-1)}(\omega_{12}\omega_{34} - \omega_{13}\omega_{24} + \omega_{14}\omega_{23} + \dots + \omega_{4n-3,4n-2}\omega_{4n-1,4n} - \omega_{4n-3,4n-1}\omega_{4n-2,4n} \\ &\quad + \omega_{4n-3,4n}\omega_{4n-2,4n-1})\Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge (d\Theta^1)^{2n} = \frac{1}{8n(2n-1)}(I_{\alpha\beta}^1 I_{\gamma\delta}^1 - I_{\alpha\gamma}^1 I_{\beta\delta}^1 + I_{\alpha\delta}^1 I_{\beta\gamma}^1) \omega_{\alpha\beta} \omega_{\gamma\delta} \times \\ &\quad \times \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge (d\Theta^1)^{2n} = \frac{1}{8n(2n-1)}(I_{\alpha\beta}^1 I_{\gamma\delta}^1 + 2I_{\alpha\delta}^1 I_{\beta\gamma}^1) \omega_{\alpha\beta} \omega_{\gamma\delta} \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge (d\Theta^1)^{2n}, \end{aligned}$$

where we used (5.1) to get the third equality in the above chain. This completes the proof of the lemma.  $\square$

**Convention 5.2.** *From now on we shall use, similarly to the CR case [22], the notation  $A \equiv B$  to designate the equivalence of the expressions  $A$  and  $B$  modulo terms that contain the torsion or the curvature or their covariant derivatives except  $R_{\alpha\beta\gamma\delta}(q)$ , as well as modulo terms of weight bigger than 4. We shall also use this notation when we omit expressions containing powers of  $\varepsilon$  that lead through the computations to powers different from  $\varepsilon^0$  and  $\varepsilon^4$ . The reason is because we know by (4.27) what kind of terms appear in the asymptotic expression of the qc Yamabe functional over the test functions.*

We continue with a lemma which is crucial for the explicit evaluation of constants.

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<sup>2</sup>Here we shall use the comma between the lower indices to separate them, not to designate the covariant differentiation. Actually, the sense of the comma will be clear from the context.

**Lemma 5.3.** *For the test function  $f^\varepsilon$  defined in Section 4 and the parabolic dilation  $\delta_\varepsilon$  in qc parabolic normal coordinates defined in Section 3 the following formulas hold:*

$$(5.2) \quad \begin{aligned} \delta_\varepsilon^*[(f^\varepsilon)^{2*} \eta^1 \wedge \eta^2 \wedge \eta^3 \wedge (d\eta^1)^{2n}] &\equiv F^{2*} [1 + \varepsilon^4 \chi(x)] \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge (d\Theta^1)^{2n}; \\ \delta_\varepsilon^*[\|\nabla f^\varepsilon\|_\eta^2 \eta^1 \wedge \eta^2 \wedge \eta^3 \wedge (d\eta^1)^{2n}] &\equiv 16(n+1)^2 [(1 + |p|^2)^2 + |w|^2]^{-2n-4} \{ [(1 + |p|^2)^2 + |w|^2] |p|^2 \\ &+ \frac{17}{720} \varepsilon^4 \sum_{i=1}^3 I_{\beta\alpha}^i I_{\xi\theta}^i R_{\delta\alpha\gamma}^\gamma(q) R_{\eta\gamma\zeta}^\theta(q) x^\beta x^\delta x^\epsilon x^\zeta x^\eta x^\xi (t^i)^2 \} [1 + \varepsilon^4 \chi(x)] \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge (d\Theta^1)^{2n}, \end{aligned}$$

where  $\chi(x)$  is a homogeneous function of order 4 defined below in (5.20).

*Proof.* We begin with the relations

$$(5.3) \quad \begin{aligned} \eta_{(2)}^i &= \Theta^i; \quad \eta_{(3)}^i = 0; \quad \eta_{(m)}^i \equiv \frac{1}{m} (t^j \omega_j^i - 2I_{\alpha\beta}^i x^\alpha \theta^\beta)_{(m)}, \quad m \geq 4; \\ \theta_{(1)}^\alpha &= \Xi^\alpha; \quad \theta_{(2)}^\alpha = 0; \quad \theta_{(m)}^\alpha \equiv \frac{1}{m} (x^\beta \omega_\beta^\alpha)_{(m)}, \quad m \geq 3; \\ \omega_{a(1)}^b &= 0; \quad \omega_{a(m)}^b \equiv \frac{1}{m} (R_{\alpha\beta a}^b x^\alpha \theta^\beta)_{(m)} \equiv \frac{1}{m} R_{\alpha\beta a}^b(q) x^\alpha \theta_{(m-1)}^\beta, \quad m \geq 2, \end{aligned}$$

which are consequences of Proposition 3.2.

We examine the homogeneous parts of certain orders. For the lowest possible order we have the formula

$$(5.4) \quad [\eta^1 \wedge \eta^2 \wedge \eta^3 \wedge (d\eta^1)^{2n}]_{(4n+6)} = \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge (d\Theta^1)^{2n}.$$

We see by a simple induction over  $m$  in (5.3) that only the even-degree homogeneous parts of  $\eta^1, \eta^2$  and  $\eta^3$  are non-zero which implies

$$(5.5) \quad [\eta^1 \wedge \eta^2 \wedge \eta^3 \wedge (d\eta^1)^{2n}]_{(4n+7)} \equiv [\eta^1 \wedge \eta^2 \wedge \eta^3 \wedge (d\eta^1)^{2n}]_{(4n+9)} \equiv 0.$$

The situation with the terms  $[\eta^1 \wedge \eta^2 \wedge \eta^3 \wedge (d\eta^1)^{2n}]_{(4n+8)}$  and  $[\eta^1 \wedge \eta^2 \wedge \eta^3 \wedge (d\eta^1)^{2n}]_{(4n+10)}$  is more complicated. We have for the first one the decomposition

$$(5.6) \quad \begin{aligned} [\eta^1 \wedge \eta^2 \wedge \eta^3 \wedge (d\eta^1)^{2n}]_{(4n+8)} &= \eta_{(2)}^1 \wedge \eta_{(2)}^2 \wedge \eta_{(4)}^3 \wedge (d\Theta^1)^{2n} + \eta_{(2)}^1 \wedge \eta_{(4)}^2 \wedge \eta_{(2)}^3 \wedge (d\Theta^1)^{2n} \\ &+ \eta_{(4)}^1 \wedge \eta_{(2)}^2 \wedge \eta_{(2)}^3 \wedge (d\Theta^1)^{2n} + 2n\eta_{(2)}^1 \wedge \eta_{(2)}^2 \wedge \eta_{(2)}^3 \wedge (d\eta^1)_{(4)} \wedge (d\Theta^1)^{2n-1}. \end{aligned}$$

We get by (5.3) and some simple calculations that

$$(5.7) \quad \eta_{(4)}^i \equiv -\frac{1}{12} I_{\alpha\beta}^i R_{\delta\zeta\gamma}^\beta(q) x^\alpha x^\gamma x^\delta dx^\zeta, \quad i = 1, 2, 3,$$

which leads to the relations

$$(5.8) \quad \eta_{(2)}^1 \wedge \eta_{(2)}^2 \wedge \eta_{(4)}^3 \wedge (d\Theta^1)^{2n} \equiv \eta_{(2)}^1 \wedge \eta_{(4)}^2 \wedge \eta_{(2)}^3 \wedge (d\Theta^1)^{2n} \equiv \eta_{(4)}^1 \wedge \eta_{(2)}^2 \wedge \eta_{(2)}^3 \wedge (d\Theta^1)^{2n} \equiv 0.$$

Furthermore, we obtain by (5.7)

$$(5.9) \quad (d\eta^i)_{(4)} = d(\eta_{(4)}^i) \equiv -\frac{1}{12} [I_{\alpha\beta}^i R_{\delta\zeta\gamma}^\beta(q) x^\gamma x^\delta + I_{\gamma\beta}^i R_{\delta\zeta\alpha}^\beta(q) x^\gamma x^\delta + I_{\delta\beta}^i R_{\alpha\zeta\gamma}^\beta(q) x^\delta x^\gamma] dx^\alpha \wedge dx^\zeta$$

which gives by straightforward calculations the formula for the trace of  $d\eta_{(4)}^1$ :

$$tr(d\eta_{(4)}^1) = -\frac{1}{2} I_{\alpha\beta}^1 (d\eta_{(4)}^1)_{\alpha\beta} \equiv -\frac{1}{24} R_{\delta\gamma}(q) x^\delta x^\gamma + \frac{n}{6} I_{\gamma}^1 \zeta_{1\delta\beta}(q) x^\gamma x^\delta - \frac{n}{6} I_{\delta}^1 \beta \sigma_{1\gamma\beta}(q) x^\gamma x^\delta,$$

where the tensors  $\zeta_{1\delta\beta}$  and  $\sigma_{1\gamma\beta}$  are defined in (2.3). The latter combined with Theorem 3.6 yields  $tr(d\eta_{(4)}^1) \equiv 0$  which together with Lemma 5.1, a) lead to  $2n\eta_{(2)}^1 \wedge \eta_{(2)}^2 \wedge \eta_{(2)}^3 \wedge (d\eta^1)_{(4)} \wedge (d\Theta^1)^{2n-1} \equiv 0$ .

Substitute the latter and (5.8) into (5.6) to get

$$(5.10) \quad [\eta^1 \wedge \eta^2 \wedge \eta^3 \wedge (d\eta^1)^{2n}]_{(4n+8)} \equiv 0.$$

We continue with the investigation of the term  $[\eta^1 \wedge \eta^2 \wedge \eta^3 \wedge (d\eta^1)^{2n}]_{(4n+10)}$ . We have the decomposition

$$(5.11) \quad \begin{aligned} [\eta^1 \wedge \eta^2 \wedge \eta^3 \wedge (d\eta^1)^{2n}]_{(4n+10)} &= \eta_{(2)}^1 \wedge \eta_{(2)}^2 \wedge \eta_{(2)}^3 \wedge [(d\eta^1)^{2n}]_{(4n+4)} \\ &+ [\eta^1 \wedge \eta^2 \wedge \eta^3]_{(8)} \wedge [(d\eta^1)^{2n}]_{(4n+2)} + [\eta^1 \wedge \eta^2 \wedge \eta^3]_{(10)} \wedge [(d\eta^1)^{2n}]_{(4n)}, \end{aligned}$$

and examine each of the terms in the right-hand side of (5.11). The first one is decomposed as follows:

$$(5.12) \quad \eta_{(2)}^1 \wedge \eta_{(2)}^2 \wedge \eta_{(2)}^3 \wedge [(d\eta^1)^{2n}]_{(4n+4)} \\ = 2n\Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge (d\eta^1)_{(6)} \wedge (d\Theta^1)^{2n-1} + n(2n-1)\Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge (d\eta_{(4)}^1)^2 \wedge (d\Theta^1)^{2n-2}.$$

To handle the first term in the right-hand side of (5.12), we use that

$$(5.13) \quad \eta_{(6)}^i \equiv -\frac{1}{360} I_{\alpha\beta}^i R_{\delta\theta\gamma}{}^\beta(q) R_{\eta\zeta}{}^\theta(q) x^\alpha x^\gamma x^\delta x^\zeta x^\eta dx^\iota, \quad i = 1, 2, 3,$$

following easily from (5.3). The formula (5.13) yields the equivalence

$$(d\eta^i)_{(6)} = d(\eta_{(6)}^i) \equiv -\frac{1}{360} (I_{\alpha\beta}^i R_{\delta\theta\gamma}{}^\beta(q) R_{\eta\sigma\zeta}{}^\theta(q) + I_{\gamma\beta}^i R_{\delta\theta\alpha}{}^\beta(q) R_{\eta\sigma\zeta}{}^\theta(q) + I_{\delta\beta}^i R_{\alpha\theta\gamma}{}^\beta(q) R_{\eta\sigma\zeta}{}^\theta(q) \\ + I_{\zeta\beta}^i R_{\delta\theta\gamma}{}^\beta(q) R_{\eta\sigma\alpha}{}^\theta(q) + I_{\eta\beta}^i R_{\delta\theta\gamma}{}^\beta(q) R_{\alpha\sigma\zeta}{}^\theta(q)) x^\gamma x^\delta x^\zeta x^\eta dx^\alpha \wedge dx^\sigma, \quad i = 1, 2, 3,$$

which together with Theorem 3.6 and some standard computations give

$$tr(d\eta_{(6)}^1) \equiv \frac{1}{720} (R_{\delta\theta\gamma}{}^\beta(q) R_{\eta\beta\zeta}{}^\theta(q) + I^{1\alpha\sigma} I_{\gamma\beta}^1 R_{\delta\theta\alpha}{}^\beta(q) R_{\eta\sigma\zeta}{}^\theta(q) + I^{1\alpha\sigma} I_{\delta\beta}^1 R_{\alpha\theta\gamma}{}^\beta(q) R_{\eta\sigma\zeta}{}^\theta(q)) x^\gamma x^\delta x^\zeta x^\eta.$$

The last formula and Lemma 5.1, a) imply

$$(5.14) \quad 2n\Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge (d\eta^1)_{(6)} \wedge (d\Theta^1)^{2n-1} \equiv \frac{1}{720} (R_{\delta\theta\gamma}{}^\beta(q) R_{\eta\beta\zeta}{}^\theta(q) + I^{1\alpha\sigma} I_{\gamma\beta}^1 R_{\delta\theta\alpha}{}^\beta(q) R_{\eta\sigma\zeta}{}^\theta(q) \\ + I^{1\alpha\sigma} I_{\delta\beta}^1 R_{\alpha\theta\gamma}{}^\beta(q) R_{\eta\sigma\zeta}{}^\theta(q)) x^\gamma x^\delta x^\zeta x^\eta \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge (d\Theta^1)^{2n}.$$

In order to manipulate the second object in the right-hand side of (5.12), we use Lemma 5.1, b), the equivalence (5.9) and Theorem 3.6. After some long but standard calculations we establish the equivalence

$$(5.15) \quad n(2n-1)\Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge (d\eta_{(4)}^1)^2 \wedge (d\Theta^1)^{2n-2} \\ \equiv -\frac{1}{576} (R_{\delta\alpha\gamma}{}^\beta(q) R_{\eta\beta\zeta}{}^\alpha(q) + 2I^{1\alpha\theta} I_{\zeta\iota}^1 R_{\delta\theta\gamma}{}^\beta(q) R_{\eta\beta\alpha}{}^\iota(q) + 2I^{1\alpha\theta} I_{\eta\iota}^1 R_{\delta\theta\gamma}{}^\beta(q) R_{\alpha\beta\zeta}{}^\iota(q) \\ + I^{1\alpha\kappa} I^{1\lambda\theta} I_{\gamma\beta}^1 I_{\zeta\iota}^1 R_{\delta\theta\alpha}{}^\beta(q) R_{\eta\kappa\lambda}{}^\iota(q) + I^{1\alpha\kappa} I^{1\lambda\theta} I_{\delta\beta}^1 I_{\eta\iota}^1 R_{\alpha\theta\gamma}{}^\beta(q) R_{\lambda\kappa\zeta}{}^\iota(q) \\ + 2I^{1\alpha\kappa} I^{1\lambda\theta} I_{\gamma\beta}^1 I_{\eta\iota}^1 R_{\delta\theta\alpha}{}^\beta(q) R_{\lambda\kappa\zeta}{}^\iota(q)) x^\gamma x^\delta x^\zeta x^\eta \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge (d\Theta^1)^{2n}.$$

We continue with the second term in the right-hand side of (5.11), which can be decomposed as follows

$$(5.16) \quad [\eta^1 \wedge \eta^2 \wedge \eta^3]_{(8)} \wedge [(d\eta^1)^{2n}]_{(4n+2)} = 2n\eta_{(4)}^1 \wedge \eta_{(2)}^2 \wedge \eta_{(2)}^3 \wedge (d\eta^1)_{(4)} \wedge (d\Theta^1)^{2n-1} \\ + 2n\eta_{(2)}^1 \wedge \eta_{(4)}^2 \wedge \eta_{(2)}^3 \wedge (d\eta^1)_{(4)} \wedge (d\Theta^1)^{2n-1} + 2n\eta_{(2)}^1 \wedge \eta_{(2)}^2 \wedge \eta_{(4)}^3 \wedge (d\eta^1)_{(4)} \wedge (d\Theta^1)^{2n-1}.$$

According to (5.7) and (5.9) the forms  $\eta_{(4)}^i$  and  $(d\eta^i)_{(4)}$  have no  $dt^j$  term which together with (5.16) imply

$$(5.17) \quad [\eta^1 \wedge \eta^2 \wedge \eta^3]_{(8)} \wedge [(d\eta^1)^{2n}]_{(4n+2)} \equiv 0.$$

We decompose the last term in the right-hand side of (5.11) in a similar manner, namely

$$[\eta^1 \wedge \eta^2 \wedge \eta^3]_{(10)} \wedge [(d\eta^1)^{2n}]_{(4n)} = \eta_{(2)}^1 \wedge \eta_{(4)}^2 \wedge \eta_{(4)}^3 \wedge (d\Theta^1)^{2n} + \eta_{(4)}^1 \wedge \eta_{(2)}^2 \wedge \eta_{(4)}^3 \wedge (d\Theta^1)^{2n} + \eta_{(4)}^1 \wedge \eta_{(4)}^2 \wedge \eta_{(2)}^3 \wedge (d\Theta^1)^{2n} \\ + \eta_{(2)}^1 \wedge \eta_{(2)}^2 \wedge \eta_{(6)}^3 \wedge (d\Theta^1)^{2n} + \eta_{(2)}^1 \wedge \eta_{(6)}^2 \wedge \eta_{(2)}^3 \wedge (d\Theta^1)^{2n} + \eta_{(6)}^1 \wedge \eta_{(2)}^2 \wedge \eta_{(2)}^3 \wedge (d\Theta^1)^{2n}.$$

Since the forms  $\eta_{(4)}^i$  and  $\eta_{(6)}^i$  do not contain  $dt^j$  term by (5.7) and (5.13), the last identity gives the relation

$$(5.18) \quad [\eta^1 \wedge \eta^2 \wedge \eta^3]_{(10)} \wedge [(d\eta^1)^{2n}]_{(4n)} \equiv 0.$$

Now we get by (5.11), (5.12), (5.14), (5.15), (5.17) and (5.18) the equivalence

$$(5.19) \quad [\eta^1 \wedge \eta^2 \wedge \eta^3 \wedge (d\eta^1)^{2n}]_{(4n+10)} \equiv \chi(x) \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge (d\Theta^1)^{2n},$$

where the homogeneous function  $\chi$  is defined by the equality

$$(5.20) \quad \chi(x) = \chi(x^1, \dots, x^{4n}) := \left( -\frac{1}{2880} R_{\delta\alpha\gamma}{}^\beta(q) R_{\eta\beta\zeta}{}^\alpha(q) - \frac{1}{480} I^{1\alpha\theta} I^1_{\zeta\iota} R_{\delta\theta\gamma}{}^\beta(q) R_{\eta\beta\alpha}{}^\iota(q) \right. \\ \left. - \frac{1}{480} I^{1\alpha\theta} I^1_{\eta\iota} R_{\delta\theta\gamma}{}^\beta(q) R_{\alpha\beta\zeta}{}^\iota(q) - \frac{1}{576} I^{1\alpha\kappa} I^{1\lambda\theta} I^1_{\gamma\beta} I^1_{\zeta\iota} R_{\delta\theta\alpha}{}^\beta(q) R_{\eta\kappa\lambda}{}^\iota(q) \right. \\ \left. - \frac{1}{576} I^{1\alpha\kappa} I^{1\lambda\theta} I^1_{\delta\beta} I^1_{\eta\iota} R_{\alpha\theta\gamma}{}^\beta(q) R_{\lambda\kappa\zeta}{}^\iota(q) - \frac{1}{288} I^{1\alpha\kappa} I^{1\lambda\theta} I^1_{\gamma\beta} I^1_{\eta\iota} R_{\delta\theta\alpha}{}^\beta(q) R_{\lambda\kappa\zeta}{}^\iota(q) \right) x^\gamma x^\delta x^\zeta x^\eta.$$

The relations (5.4), (5.5), (5.10) and (5.19) lead to the equivalence<sup>3</sup>

$$\eta^1 \wedge \eta^2 \wedge \eta^3 \wedge (d\eta^1)^{2n} \equiv [1 + \chi(x)] \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge (d\Theta^1)^{2n},$$

which in turn gives

$$(5.21) \quad \delta_\varepsilon^*[\eta^1 \wedge \eta^2 \wedge \eta^3 \wedge (d\eta^1)^{2n}] \equiv \varepsilon^{4n+6} [1 + \varepsilon^4 \chi(x)] \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge (d\Theta^1)^{2n}.$$

It follows by the very definition of the test function  $f^\varepsilon$  that  $\delta_\varepsilon^*(f^\varepsilon)^{2^*} = \varepsilon^{-4n-6} \delta_\varepsilon^*(\psi^{2^*}) F^{2^*}$ , which together with (5.21) give the first formula in (5.2). We omit the multiplier  $\delta_\varepsilon^*(\psi^{2^*})$  since we know from (4.32) and (4.35) that it appears only in the  $O(\varepsilon^5)$ -part of the denominator of the asymptotic expansion (4.27).

In order to prove the second formula in (5.2), we need to find the effect of the parabolic dilation change of the variables on the squared norm

$$(5.22) \quad |\nabla f^\varepsilon|_\eta^2 := \sum_{\alpha=1}^{4n} (\xi_\alpha f^\varepsilon)^2 = \sum_{\alpha=1}^{4n} (s_\alpha^a X_a f^\varepsilon) (s_\alpha^b X_b f^\varepsilon).$$

We continue with the computations we commenced in Lemma 4.2. We have

$$(5.23) \quad s_{\alpha(1)}^\beta = - \sum_{i \geq 2} s_{\alpha(o(a)-i)}^a \theta_{(1+i)}^\beta(X_a) = 0,$$

where the first identity follows from (4.2) and the second one—from (4.1). For the term  $s_{\alpha(2)}^\beta$  we obtain

$$(5.24) \quad s_{\alpha(2)}^\beta = - \sum_{i \geq 2} s_{\alpha(2-i)}^\gamma \theta_{(1+i)}^\beta(X_\gamma) - \sum_{i \geq 2} s_{\alpha(3-i)}^{\tilde{\alpha}} \theta_{(1+i)}^\beta(X_{\tilde{\alpha}}) = -\theta_{(3)}^\beta(X_\alpha) \equiv -\frac{1}{6} R_{\delta\alpha\gamma}{}^\beta(q) x^\gamma x^\delta,$$

in which we utilized (4.2) to get the first relation, whereas the second one is obtained by (4.1) and the last equivalence is a result of a repeated application of the relations in (5.3). In the same spirit we get

$$(5.25) \quad s_{\alpha(2)}^{\tilde{\alpha}} = 0, \quad s_{\alpha(3)}^\beta \equiv 0.$$

Note that the first one is obtained with the help of (4.2) and (4.1), while we used (4.2), (4.1), (5.23), the first relation in (5.25) and (5.3) to establish the second one. Regarding  $s_{\alpha(3)}^{\tilde{\alpha}}$ , we obtain similarly that

$$(5.26) \quad s_{\alpha(3)}^{\tilde{\alpha}} \equiv \frac{1}{12} I^{\tilde{\alpha}}_{\beta\gamma} R_{\theta\alpha\delta}{}^\gamma(q) x^\beta x^\delta x^\theta,$$

where we set  $I^{\tilde{\alpha}}_{\beta\gamma} := I^{\tilde{\alpha}-4n}_{\beta\gamma}$  and used (4.2), (4.1) and (5.3).

We get for the term  $s_{\alpha(4)}^\beta$  the following chain of relations

$$(5.27) \quad s_{\alpha(4)}^\beta = - \sum_{i \geq 2} s_{\alpha(4-i)}^\gamma \theta_{(1+i)}^\beta(X_\gamma) - \sum_{i \geq 2} s_{\alpha(5-i)}^{\tilde{\alpha}} \theta_{(1+i)}^\beta(X_{\tilde{\alpha}}) \equiv \frac{7}{360} R_{\delta\alpha\theta}{}^\gamma(q) R_{\eta\gamma\zeta}{}^\beta(q) x^\delta x^\zeta x^\eta x^\theta,$$

where we took into account (4.2) to obtain the first equality, while the second one is a result of (4.1), (5.23), (5.24), (5.26) and a repeated application of (5.3).

In a similar way, we establish using (4.2), (4.1), (5.23), (5.25) and (5.3) that

$$(5.28) \quad s_{\alpha(4)}^{\tilde{\alpha}} \equiv 0.$$

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<sup>3</sup>Note that we consider the homogeneous parts of  $\eta^1 \wedge \eta^2 \wedge \eta^3 \wedge (d\eta^1)^{2n}$  up to order  $4n + 10$  since the higher order terms belong to  $O(\varepsilon^5)$  in the expressions after the parabolic dilation change of the variables.

Similarly, using (4.2), (4.1), (5.23), (5.24), (5.26) and (5.3), we get

$$(5.29) \quad s_{\alpha(5)}^{\tilde{\alpha}} \equiv -\frac{1}{90} I_{\tilde{\alpha}\theta\zeta}^{\tilde{\alpha}} R_{\delta\alpha\gamma}^{\beta}(q) R_{\xi\beta\eta}^{\zeta}(q) x^{\gamma} x^{\delta} x^{\eta} x^{\theta} x^{\xi}.$$

To find the effect of the parabolic dilation change of the variables on the squared norm (5.22) we describe the result of this change on the functions  $X_{\alpha}F^{\varepsilon}$  and  $X_{\tilde{\alpha}}F^{\varepsilon}$ . We obtain with some standard calculations that

$$(5.30) \quad \begin{aligned} \delta_{\varepsilon}^*(X_{\alpha}F^{\varepsilon}) &= -4(n+1)\varepsilon^{-2n-3}[(1+|p|^2)^2 + |w|^2]^{-n-2}[(1+|p|^2)x^{\alpha} + \sum_{i=1}^3 I_{\beta\alpha}^i x^{\beta} t^i]; \\ \delta_{\varepsilon}^*(X_{\tilde{\alpha}}F^{\varepsilon}) &= -4(n+1)\varepsilon^{-2n-4}[(1+|p|^2)^2 + |w|^2]^{-n-2}t^{\tilde{\alpha}}, \end{aligned}$$

where in the right-hand side of the last formula we set  $t^{\tilde{\alpha}} := t^{\tilde{\alpha}-4n}$ . Now we have

$$(5.31) \quad \begin{aligned} \delta_{\varepsilon}^*(\xi_{\alpha}f^{\varepsilon}) &\equiv \delta_{\varepsilon}^*(s_{\alpha}^{\beta}X_{\beta}F^{\varepsilon} + s_{\alpha}^{\tilde{\alpha}}X_{\tilde{\alpha}}F^{\varepsilon}) \\ &\equiv \delta_{\varepsilon}^*[(s_{\alpha(0)}^{\beta} + s_{\alpha(1)}^{\beta} + s_{\alpha(2)}^{\beta} + s_{\alpha(3)}^{\beta} + s_{\alpha(4)}^{\beta})X_{\beta}F^{\varepsilon} + (s_{\alpha(0)}^{\tilde{\alpha}} + s_{\alpha(1)}^{\tilde{\alpha}} + s_{\alpha(2)}^{\tilde{\alpha}} + s_{\alpha(3)}^{\tilde{\alpha}} + s_{\alpha(4)}^{\tilde{\alpha}} + s_{\alpha(5)}^{\tilde{\alpha}})X_{\tilde{\alpha}}F^{\varepsilon}] \\ &\equiv \delta_{\varepsilon}^*(X_{\alpha}F^{\varepsilon} - \frac{1}{6}R_{\delta\alpha\gamma}^{\beta}(q)x^{\gamma}x^{\delta}X_{\beta}F^{\varepsilon} + \frac{7}{360}R_{\delta\alpha\theta}^{\gamma}(q)R_{\eta\gamma\zeta}^{\beta}(q)x^{\delta}x^{\zeta}x^{\eta}x^{\theta}X_{\beta}F^{\varepsilon} + \frac{1}{12}I_{\beta\gamma}^{\tilde{\alpha}}R_{\theta\alpha\delta}^{\gamma}(q)x^{\beta}x^{\delta}x^{\theta}X_{\tilde{\alpha}}F^{\varepsilon} \\ &\quad - \frac{1}{90}I_{\tilde{\alpha}\theta\zeta}^{\tilde{\alpha}}R_{\delta\alpha\gamma}^{\beta}(q)R_{\xi\beta\eta}^{\zeta}(q)x^{\gamma}x^{\delta}x^{\eta}x^{\theta}x^{\xi}X_{\tilde{\alpha}}F^{\varepsilon}) = -4(n+1)\varepsilon^{-2n-3}[(1+|p|^2)^2 + |w|^2]^{-n-2}[(1+|p|^2)x^{\alpha} \\ &\quad + \sum_{i=1}^3 I_{\beta\alpha}^i x^{\beta} t^i] - \frac{1}{12}\varepsilon^2 \sum_{i=1}^3 I_{\eta\beta}^i R_{\delta\alpha\gamma}^{\beta}(q)x^{\gamma}x^{\delta}x^{\eta}t^i + \frac{1}{120}\varepsilon^4 \sum_{i=1}^3 I_{\xi\beta}^i R_{\delta\alpha\theta}^{\gamma}(q)R_{\eta\gamma\zeta}^{\beta}(q)x^{\delta}x^{\zeta}x^{\eta}x^{\theta}x^{\xi}t^i). \end{aligned}$$

In order to get the first equivalence in (5.31) we use  $f^{\varepsilon} = \psi F^{\varepsilon} \equiv F^{\varepsilon}$  since (4.36) shows that the function  $\psi$  contributes only in the integral  $I_2$  which is  $O(\varepsilon^5)$  according to (4.48). The third equivalence is obtained with the help of the relations (4.1), (5.23), (5.24), (5.25), (5.26), (5.27), (5.28) and (5.29). Finally, the last relation in (5.31) is a result of (5.30) and the second identity in (3.6). The formula (5.31) together with (3.6) and some standard calculations lead to

$$\begin{aligned} \delta_{\varepsilon}^*|\nabla f^{\varepsilon}|_{\eta}^2 &\equiv 16(n+1)^2\varepsilon^{-4n-6}[(1+|p|^2)^2 + |w|^2]^{-2n-4} \times \\ &\quad \times \{[(1+|p|^2)^2 + |w|^2]|p|^2 + \frac{17}{720}\varepsilon^4 \sum_{i=1}^3 I_{\beta\alpha}^i I_{\xi\theta}^i R_{\delta\alpha\iota}^{\gamma}(q)R_{\eta\gamma\zeta}^{\theta}(q)x^{\beta}x^{\delta}x^{\zeta}x^{\eta}x^{\iota}x^{\xi}(t^i)^2\}, \end{aligned}$$

where we omitted the terms that contain powers of  $\varepsilon$  different from 0 and 4. Moreover,  $t^i$  must appear only in even powers due to the integration reasons. The last formula together with (5.21) proves the second formula in (5.2) which completes the proof of the lemma.  $\square$

We continue with a fact from the multivariable calculus. We recall that the natural volume form on  $\mathbb{R}^{4n}$  is  $dx := dx^1 \wedge \dots \wedge dx^{4n}$ . The polar change  $(x^1, \dots, x^{4n}) = (r\zeta^1, \dots, r\zeta^{4n})$ ,  $\zeta := (\zeta^1, \dots, \zeta^{4n}) \in S^{4n-1}$ ,  $r > 0$ , leads to the representation  $dx = r^{4n-1}dr \wedge d\omega$ , where  $d\omega$  is a volume form on  $S^{4n-1}$ . We have

**Proposition 5.4.** *If  $\alpha_1, \dots, \alpha_{4n}$  are some non-negative integers then the next formulas hold:*

$$(5.32) \quad \begin{aligned} \int_{S^{4n-1}} (\zeta^1)^{\alpha_1} \dots (\zeta^{4n})^{\alpha_{4n}} d\omega &= 0, & \text{if some } \alpha_s \text{ is odd;} \\ \int_{S^{4n-1}} (\zeta^1)^{\alpha_1} \dots (\zeta^{4n})^{\alpha_{4n}} d\omega &= \frac{\alpha_1! \dots \alpha_{4n}! \pi^{2n}}{2^{\alpha_1 + \dots + \alpha_{4n} - 1} \left(\frac{\alpha_1}{2}\right)! \dots \left(\frac{\alpha_{4n}}{2}\right)! \left(\frac{\alpha_1 + \dots + \alpha_{4n} + 4n - 2}{2}\right)!}, & \text{if any } \alpha_s \text{ is even.} \end{aligned}$$

*Proof.* For the Gamma function  $\Gamma(t) := \int_0^{\infty} s^{t-1} e^{-s} ds$  one gets after changing  $s = u^2$ ,  $u \in (0, \infty)$ ,

$$(5.33) \quad \Gamma(t) = 2 \int_0^{\infty} u^{2t-1} e^{-u^2} du.$$

For the integral  $I := \int_{\mathbb{R}^{4n}} (x^1)^{\alpha_1} \dots (x^{4n})^{\alpha_{4n}} e^{-(x^1)^2 - \dots - (x^{4n})^2} dx$  we have as an application of (5.33)

$$(5.34) \quad I = \int_{-\infty}^{\infty} (x^1)^{\alpha_1} e^{-(x^1)^2} dx^1 \dots \int_{-\infty}^{\infty} (x^{4n})^{\alpha_{4n}} e^{-(x^{4n})^2} dx^{4n} = \begin{cases} 0, & \text{if some } \alpha_s \text{ is odd;} \\ \prod_{s=1}^{4n} \Gamma(\beta_s), & \text{if any } \alpha_s \text{ is even,} \end{cases}$$



where  $\beta_s := \frac{1}{2}(\alpha_s + 1)$ ,  $s = 1, \dots, 4n$ . On the other hand the polar change and (5.33) yield

$$(5.35) \quad I = \int_{S^{4n-1}} (\zeta^1)^{\alpha_1} \dots (\zeta^{4n})^{\alpha_{4n}} d\omega \int_0^\infty r^{\alpha_1 + \dots + \alpha_{4n} + 4n-1} e^{-r^2} dr \\ = \frac{1}{2} \Gamma(\beta_1 + \dots + \beta_{4n}) \int_{S^{4n-1}} (\zeta^1)^{\alpha_1} \dots (\zeta^{4n})^{\alpha_{4n}} d\omega.$$

We compare (5.34) and (5.35) to get the formula

$$(5.36) \quad \int_{S^{4n-1}} (\zeta^1)^{\alpha_1} \dots (\zeta^{4n})^{\alpha_{4n}} d\omega = \begin{cases} 0, & \text{if some } \alpha_s \text{ is odd;} \\ \frac{2\Gamma(\beta_1) \dots \Gamma(\beta_{4n})}{\Gamma(\beta_1 + \dots + \beta_{4n})}, & \text{if any } \alpha_s \text{ is even.} \end{cases}$$

Now, if any  $\alpha_s$  is even, the well-known formulas giving the values of the Gamma function for positive integer and half-integer arguments,

$$(5.37) \quad \Gamma(n) = (n-1)!, \quad \Gamma(n + \frac{1}{2}) = \frac{(2n-1)!!}{2^n} \sqrt{\pi},$$

imply  $\Gamma(\beta_s) = \frac{\alpha_s! \sqrt{\pi}}{(\frac{\alpha_s}{2})! 2^{\alpha_s}}$  and  $\Gamma(\beta_1 + \dots + \beta_{4n}) = \left( \frac{\alpha_1 + \dots + \alpha_{4n} + 4n - 2}{2} \right)!$  which inserted into (5.36) give (5.32).  $\square$

The last auxiliary result that help us to prove the main claim of this section is

**Corollary 5.5.** *The following formulas for integration on  $S^{4n-1}$  hold:*

$$(5.38) \quad \begin{aligned} \int_{S^{4n-1}} R_{\delta\alpha\gamma}^\beta(q) R_{\eta\beta\xi}^\alpha(q) \zeta^\gamma \zeta^\delta \zeta^\xi \zeta^\eta d\omega &= \frac{3\pi^{2n}}{4(2n+1)!} \|W^{qc}\|^2; \\ \int_{S^{4n-1}} I^{1\alpha\theta} I_{\xi\iota}^1 R_{\delta\theta\gamma}^\beta(q) R_{\eta\beta\alpha}^\iota(q) \zeta^\gamma \zeta^\delta \zeta^\xi \zeta^\eta d\omega &= \frac{3\pi^{2n}}{4(2n+1)!} \|W^{qc}\|^2; \\ \int_{S^{4n-1}} I^{1\alpha\theta} I_{\eta\iota}^1 R_{\delta\theta\gamma}^\beta(q) R_{\alpha\beta\xi}^\iota(q) \zeta^\gamma \zeta^\delta \zeta^\xi \zeta^\eta d\omega &= \frac{\pi^{2n}}{2(2n+1)!} \|W^{qc}\|^2; \\ \int_{S^{4n-1}} I^{1\alpha\kappa} I^{1\lambda\theta} I_{\gamma\beta}^1 I_{\xi\iota}^1 R_{\delta\theta\alpha}^\beta(q) R_{\eta\kappa\lambda}^\iota(q) \zeta^\gamma \zeta^\delta \zeta^\xi \zeta^\eta d\omega &= \frac{3\pi^{2n}}{4(2n+1)!} \|W^{qc}\|^2; \\ \int_{S^{4n-1}} I^{1\alpha\kappa} I^{1\lambda\theta} I_{\delta\beta}^1 I_{\eta\iota}^1 R_{\alpha\theta\gamma}^\beta(q) R_{\lambda\kappa\xi}^\iota(q) \zeta^\gamma \zeta^\delta \zeta^\xi \zeta^\eta d\omega &= \frac{\pi^{2n}}{(2n+1)!} \|W^{qc}\|^2; \\ \int_{S^{4n-1}} I^{1\alpha\kappa} I^{1\lambda\theta} I_{\gamma\beta}^1 I_{\eta\iota}^1 R_{\delta\theta\alpha}^\beta(q) R_{\lambda\kappa\xi}^\iota(q) \zeta^\gamma \zeta^\delta \zeta^\xi \zeta^\eta d\omega &= \frac{\pi^{2n}}{2(2n+1)!} \|W^{qc}\|^2; \\ \int_{S^{4n-1}} I_{\beta\alpha}^i I_{\kappa\theta}^i R_{\delta\alpha\iota}^\gamma(q) R_{\eta\gamma\xi}^\theta(q) \zeta^\beta \zeta^\delta \zeta^\xi \zeta^\eta \zeta^\iota \zeta^\kappa d\omega &= \frac{\pi^{2n}}{(2n+2)!} \|W^{qc}\|^2, \quad \text{for a fixed } i, \end{aligned}$$

where  $\|W^{qc}\|^2$  denotes the squared norm of the qc conformal curvature at  $q$ .

*Proof.* The curvature identities (3.6) imply

$$(5.39) \quad \|W^{qc}\|^2 = R_{\alpha\beta\gamma\delta}(q) R^{\alpha\beta\gamma\delta}(q) = 2R_{\alpha\beta\gamma\delta}(q) R^{\alpha\gamma\beta\delta}(q).$$

We give a detail proof of the fourth formula in (5.38) since the proof of the others is very similar.

Denote the integral that stays in the left-hand side of the formula by  $\mathcal{I}$  then we have the decomposition

$$(5.40) \quad \mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3,$$

where each of the integrals  $\mathcal{I}_1, \mathcal{I}_2$  and  $\mathcal{I}_3$  corresponds to one of the cases considered below.<sup>4</sup>

*Case 1:*  $\gamma = \delta, \xi = \eta$ . We have

$$(5.41) \quad \mathcal{I}_1 = I^{1\alpha\kappa} I^{1\lambda\theta} I_{\gamma\beta}^1 I_{\xi\iota}^1 R_{\gamma\theta\alpha}^\beta(q) R_{\xi\kappa\lambda}^\iota(q) \frac{\pi^{2n}}{2(2n+1)!} = 16n^2 I^{1\alpha\kappa} I^{1\lambda\theta} \zeta_{1\theta\alpha}(q) \zeta_{1\kappa\lambda}(q) \frac{\pi^{2n}}{2(2n+1)!} = 0,$$

where we used (5.32) to get the first identity and Theorem 3.6 to obtain the third one.

<sup>4</sup>Note that the intersection of these cases is not the empty set. They intersect in the case when  $\gamma = \delta = \xi = \eta$ , the corresponding integral of which we divide into three equals parts and include them in the integrals  $\mathcal{I}_1, \mathcal{I}_2$  and  $\mathcal{I}_3$ .

Case 2:  $\gamma = \xi, \delta = \eta$ . In this case we have

$$\begin{aligned}
 (5.42) \quad J_2 &= I^{1\alpha\kappa} I^{1\lambda\theta} \underbrace{I^1_{\gamma\beta} I^1_{\gamma\iota}}_{\delta_{\beta\iota}} R_{\delta\theta\alpha}{}^\beta(q) R_{\delta\kappa\lambda}{}^\iota(q) \frac{\pi^{2n}}{2(2n+1)!} = I^{1\alpha\kappa} I^{1\lambda\theta} R_{\delta\theta\alpha}{}^\beta(q) R_{\delta\kappa\lambda}{}^\beta(q) \frac{\pi^{2n}}{2(2n+1)!} \\
 &= \underbrace{I^{1\alpha\beta} I^{1\lambda\beta}}_{\delta_{\alpha\lambda}} R_{\delta\theta\alpha}{}^\kappa(q) R_{\delta\kappa\lambda}{}^\theta(q) \frac{\pi^{2n}}{2(2n+1)!} = R_{\delta\theta\kappa\alpha}(q) R^{\delta\kappa\theta\alpha}(q) \frac{\pi^{2n}}{2(2n+1)!} = \frac{\pi^{2n}}{4(2n+1)!} \|W^{qc}\|^2,
 \end{aligned}$$

where we used (5.32) to get the first identity and (3.6) to obtain the third and the fourth one. The fifth identity follows from the second relation in (5.39).

Case 3:  $\gamma = \eta, \delta = \xi$ . We establish in this situation

$$\begin{aligned}
 (5.43) \quad J_3 &= I^{1\alpha\kappa} I^{1\lambda\theta} I^1_{\gamma\beta} I^1_{\delta\iota} R_{\delta\theta\alpha}{}^\beta(q) R_{\gamma\kappa\lambda}{}^\iota(q) \frac{\pi^{2n}}{2(2n+1)!} = \underbrace{I^{1\alpha\beta} I^1_{\gamma\beta}}_{\delta_{\alpha\gamma}} \underbrace{I^{1\lambda\iota} I^1_{\delta\iota}}_{\delta_{\lambda\delta}} R_{\delta\theta\alpha}{}^\kappa(q) R_{\gamma\kappa\lambda}{}^\theta(q) \frac{\pi^{2n}}{2(2n+1)!} \\
 &= R_{\delta\theta\alpha}{}^\kappa(q) R_{\alpha\kappa\delta}{}^\theta(q) \frac{\pi^{2n}}{2(2n+1)!} = R_{\alpha\kappa\delta\theta}(q) R^{\alpha\kappa\delta\theta}(q) \frac{\pi^{2n}}{2(2n+1)!} = \frac{\pi^{2n}}{2(2n+1)!} \|W^{qc}\|^2,
 \end{aligned}$$

where we used (5.32), the fifth and the fourth equalities in (3.6) and the first identity in (5.39) to get the first, the second, the fourth and the fifth identity, respectively.

Now, we substitute (5.41), (5.42) and (5.43) in (5.40) to obtain the desired formula.

Note that in order to obtain the seventh formula in (5.38) we consider fifteen cases in a similar way.  $\square$

Now we are ready to prove the main result of this section:

**Theorem 5.6.** *Let  $(M, \eta)$  be a quaternionic contact manifold of dimension  $4n+3$  with a qc contact form  $\eta$  normalized at a fixed point  $q \in M$  according to Theorem 3.6. Then the qc Yamabe functional (1.1) over the test functions  $f^\varepsilon$  defined near  $q$  in Section 4 has the asymptotic expansion (1.2).*

*Proof.* We begin with the remark that the natural volume form  $Vol_\Theta$  on the Heisenberg group  $\mathbf{G}(\mathbb{H})$  can be expressed in the terms of the qc parabolic normal coordinates as follows

$$(5.44) \quad Vol_\Theta = \frac{(2n)!}{8} dt^1 \wedge dt^2 \wedge dt^3 \wedge dx^1 \wedge \dots \wedge dx^{4n}.$$

Our first aim is to investigate the numerator of (4.28). We get consecutively

$$\begin{aligned}
(5.45) \quad & \int_M \left[ 4 \frac{n+2}{n+1} |\nabla f^\varepsilon|_\eta^2 + S(f^\varepsilon)^2 \right] Vol_\eta = 4 \frac{n+2}{n+1} \int_M \delta_\varepsilon^* (|\nabla f^\varepsilon|_\eta^2 Vol_\eta) + O(\varepsilon^5) \\
& \equiv 64(n+1)(n+2) \int_{\mathbf{G}(\mathbb{H})} [(1+|p|^2)^2 + |w|^2]^{-2n-4} \{ [(1+|p|^2)^2 + |w|^2] |p|^2 \\
& \quad + \underbrace{\frac{17}{720} \varepsilon^4 \sum_{i=1}^3 I_{\beta\alpha}^i I_{\xi\theta}^i R_{\delta\alpha\iota}^\gamma(q) R_{\eta\gamma\zeta}^\theta(q) x^\beta x^\delta x^\iota x^\zeta x^\eta x^\varepsilon (t^i)^2}_{:=\tilde{\chi}(x,t)} \} [1 + \varepsilon^4 \chi(x)] Vol_\Theta \\
& \equiv 8(n+1)(n+2)(2n)! \int_{\mathbb{R}^{4n}} \int_{\mathbb{R}^3} [(1+|p|^2)^2 + |w|^2]^{-2n-4} \{ [(1+|p|^2)^2 + |w|^2] |p|^2 \\
& \quad + \{ [(1+|p|^2)^2 + |w|^2] |p|^2 \chi(x) + \frac{17}{720} \tilde{\chi}(x,t) \} \varepsilon^4 \} dt^1 \wedge dt^2 \wedge dt^3 \wedge dx^1 \wedge \dots \wedge dx^{4n} \\
& = 8(n+1)(n+2)(2n)! \int_{S^{4n-1}} \int_0^\infty \int_{\mathbb{R}^3} [(1+r^2)^2 + (t^1)^2 + (t^2)^2 + (t^3)^2]^{-2n-3} r^{4n+1} dt^1 \wedge dt^2 \wedge dt^3 \wedge dr \wedge d\omega \\
& \quad + \varepsilon^4 8(n+1)(n+2)(2n)! \int_{S^{4n-1}} \int_0^\infty \int_{\mathbb{R}^3} [(1+r^2)^2 + (t^1)^2 + (t^2)^2 + (t^3)^2]^{-2n-3} r^{4n+5} \chi(\zeta) dt^1 \wedge dt^2 \wedge dt^3 \wedge dr \wedge d\omega \\
& \quad + \varepsilon^4 \frac{17}{90} (n+1)(n+2)(2n)! \int_{S^{4n-1}} \int_0^\infty \int_{\mathbb{R}^3} [(1+r^2)^2 + (t^1)^2 + (t^2)^2 + (t^3)^2]^{-2n-4} r^{4n+5} \tilde{\chi}(\zeta, t) dt^1 \wedge dt^2 \wedge dt^3 \wedge dr \wedge d\omega \\
& \quad =: \mathcal{S}_1 + (\mathcal{S}_2 + \mathcal{S}_3) \varepsilon^4,
\end{aligned}$$

where we used the parabolic dilation change of the variables and (4.56) to get the first relation and the second equivalence in (5.2) to obtain the second one. The third equivalence is a result of (5.44), whereas the fourth one is established after the polar change in the  $x$  variable. Now we are going to calculate explicitly the integrals  $\mathcal{S}_1, \mathcal{S}_2$  and  $\mathcal{S}_3$  that appear in the formula above. We recall firstly the formula [22, (5.4)]

$$(5.46) \quad \int_0^\infty b^\gamma (a^2 + b^2)^{-\alpha/2} db = \frac{\Gamma((\gamma+1)/2) \Gamma((\alpha-\gamma-1)/2)}{2\Gamma(\alpha/2)} a^{\gamma-\alpha+1},$$

where  $a, b \in \mathbb{R}^+$  and  $\alpha, \gamma \in \mathbb{R}$ ,  $\alpha - \gamma - 1 > 0$ ,  $\gamma + 1 > 0$ .

We obtain for  $\mathcal{S}_1$  :

$$\begin{aligned}
(5.47) \quad \mathcal{S}_1 &= 32n(n+1)(n+2)\pi^{2n} \int_0^\infty \int_{\mathbb{R}^3} [(1+r^2)^2 + (t^1)^2 + (t^2)^2 + (t^3)^2]^{-2n-3} r^{4n+1} dt^1 dt^2 dt^3 dr \\
&= \frac{2n(n+2)\pi^{2n+2}}{4^{2n}(2n+1)},
\end{aligned}$$

where we applied (5.32) to get the first equality, while the second one is obtained by (5.37) and a fourfold application of (5.46). More concretely, we integrate at the first stage over the variable  $t^1$  and set  $a := [(1+r^2)^2 + (t^2)^2 + (t^3)^2]^{\frac{1}{2}}$ ,  $b := t^1$  which reduce the four-fold integral to a three-fold one and s.o.

For  $\mathcal{S}_2$ , we get by (5.20) and the first six equalities in (5.38) that

$$(5.48) \quad \int_{S^{4n-1}} \chi(\zeta) d\omega = \frac{-11\pi^{2n}}{1440(2n+1)!} \|W^{qc}\|^2.$$

We obtain with the help of (5.48) in a similar way as (5.47) the following representation of  $\mathcal{S}_2$  :

$$(5.49) \quad \mathcal{S}_2 = \frac{-11(n+1)(n+2)\pi^{2n+2}}{45 \cdot 4^{2n+3} n(2n+1)^2} \|W^{qc}\|^2.$$

We use the seventh formula in (5.38) to get after certain calculations similarly to (5.47) and (5.49) that

$$(5.50) \quad \mathcal{S}_3 = \frac{17(n+2)\pi^{2n+2}}{30 \cdot 4^{2n+3} n(2n+1)^2(2n+3)} \|W^{qc}\|^2.$$

Now we substitute (5.47), (5.49) and (5.50) into (5.45) to obtain

$$(5.51) \quad \int_M \left[ 4 \frac{n+2}{n+1} |\nabla f^\varepsilon|_\eta^2 + S(f^\varepsilon)^2 \right] Vol_\eta \\ = \frac{2n(n+2)\pi^{2n+2}}{4^{2n}(2n+1)} + \frac{(n+2)(-44n^2 - 110n - 15)\pi^{2n+2}}{90.4^{2n+3}n(2n+1)^2(2n+3)} \|W^{qc}\|^2 \varepsilon^4 + O(\varepsilon^5).$$

Our next goal is to calculate the integral that stays in the denominator of (4.28). We have

$$(5.52) \quad \int_M (f^\varepsilon)^{2*} Vol_\eta = \int_M \delta_\varepsilon^* [(f^\varepsilon)^{2*} Vol_\eta] \equiv \int_{\mathbf{G}(\mathbb{H})} F^{2*} [1 + \varepsilon^4 \chi(x)] Vol_\Theta \\ = \frac{(2n)!}{8} \int_{\mathbb{R}^{4n}} \int_{\mathbb{R}^3} [(1 + |p|^2)^2 + |w|^2]^{-2n-3} [1 + \varepsilon^4 \chi(x)] dt^1 \wedge dt^2 \wedge dt^3 \wedge dx^1 \wedge \dots \wedge dx^{4n} \\ = \frac{(2n)!}{8} \int_{S^{4n-1}} \int_0^\infty \int_{\mathbb{R}^3} [(1 + r^2)^2 + (t^1)^2 + (t^2)^2 + (t^3)^2]^{-2n-3} r^{4n-1} dt^1 \wedge dt^2 \wedge dt^3 \wedge dr \wedge d\omega \\ + \varepsilon^4 \frac{(2n)!}{8} \int_{S^{4n-1}} \int_0^\infty \int_{\mathbb{R}^3} [(1 + r^2)^2 + (t^1)^2 + (t^2)^2 + (t^3)^2]^{-2n-3} r^{4n+3} \chi(\zeta) dt^1 \wedge dt^2 \wedge dt^3 \wedge dr \wedge d\omega \\ =: \tilde{S}_1 + \tilde{S}_2 \varepsilon^4,$$

where we made change of the variables to get the first relation and applied the first equivalence from (5.2) to obtain the second one. Furthermore, we used (5.44) to establish the third identity in (5.52), whereas the fourth one is an effect of the polar change in the  $x$  variable.

The calculations of  $\tilde{S}_1$  and  $\tilde{S}_2$  are analogous to those we made in obtaining (5.47), (5.49) and (5.50). We get as a result the formulas

$$\tilde{S}_1 = \frac{\pi^{2n+2}}{2.4^{2n+2}(2n+1)} \quad \text{and} \quad \tilde{S}_2 = \frac{-11\pi^{2n+2}}{90.4^{2n+5}(2n+1)^2(2n+2)} \|W^{qc}\|^2$$

which we substitute in (5.52) to establish the identity

$$(5.53) \quad \int_M (f^\varepsilon)^{2*} Vol_\eta = \frac{\pi^{2n+2}}{2.4^{2n+2}(2n+1)} + \frac{-11\pi^{2n+2}}{90.4^{2n+5}(2n+1)^2(2n+2)} \|W^{qc}\|^2 \varepsilon^4 + O(\varepsilon^5).$$

Now we insert (5.51) and (5.53) into (4.28) to get

$$\Upsilon_\eta(f^\varepsilon) = \left[ \frac{2n(n+2)\pi^{2n+2}}{4^{2n}(2n+1)} + \frac{(n+2)(-44n^2 - 110n - 15)\pi^{2n+2}}{90.4^{2n+3}n(2n+1)^2(2n+3)} \|W^{qc}\|^2 \varepsilon^4 + O(\varepsilon^5) \right] \times \\ \times \left[ \frac{\pi^{2n+2}}{2.4^{2n+2}(2n+1)} + \frac{-11\pi^{2n+2}}{90.4^{2n+5}(2n+1)^2(2n+2)} \|W^{qc}\|^2 \varepsilon^4 + O(\varepsilon^5) \right]^{-2/2*} \\ = \frac{\pi^{-\frac{(2n+2)^2}{2n+3}}}{[2.4^{2n+2}(2n+1)]^{-\frac{2n+2}{2n+3}}} \left[ \frac{2n(n+2)\pi^{2n+2}}{4^{2n}(2n+1)} + \frac{(n+2)(-44n^2 - 110n - 15)\pi^{2n+2}}{90.4^{2n+3}n(2n+1)^2(2n+3)} \|W^{qc}\|^2 \varepsilon^4 + O(\varepsilon^5) \right] \times \\ \times \left[ 1 + \frac{11\|W^{qc}\|^2 \varepsilon^4}{2880(2n+1)(2n+3)} + O(\varepsilon^5) \right] = \Lambda(1 - c(n)\|W^{qc}\|^2 \varepsilon^4) + O(\varepsilon^5),$$

where  $c(n) := \frac{22n+3}{2304n^2(2n+1)(2n+3)}$ . The theorem is proved.  $\square$

## REFERENCES

- [1] Aubin, Th., *Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl. (9) **55** (1976), no. 3, 269–296. [3](#)
- [2] Bahri, A., *Proof of the Yamabe conjecture for locally conformally flat manifolds*, Non. Analysis, Theory, Methods and Appli., 20 (10) (1993), 1261–1278. [3](#)
- [3] Bahri, A., Brezis, H., *Nonlinear elliptic equations*, in Topics in Geometry in memory of Joseph D'Atri, Simon Gindikin editor, Birkhauser, Boston-Basel-Berlin, (1996), 1–100. [3](#)
- [4] Biquard, O., *Métriques d'Einstein asymptotiquement symétriques*, Astérisque **265** (2000). [1](#), [2](#), [4](#), [5](#), [6](#)
- [5] Biquard, O., *Quaternionic contact structures*, Quaternionic structures in mathematics and physics (Rome, 1999), 23–30 (electronic), Univ. Studi Roma "La Sapienza", Roma, 1999.

- [6] Chern, S.S. & Moser, J., *Real hypersurfaces in complex manifolds*, Acta Math. **133** (1974), 219–271. [7](#)
- [7] Duchemin, D., *Quaternionic contact structures in dimension 7*, Ann. Inst. Fourier (Grenoble) **56** (2006), no. 4, 851–885. [4](#), [5](#)
- [8] Fefferman, C., & Graham, C.R., *Conformal invariants*. The mathematical heritage of Élie Cartan (Lyon, 1984). Astérisque 1985, Numéro Hors Série, 95–116. [2](#)
- [9] Folland, G.B., *Subelliptic estimates and function spaces on nilpotent Lie groups*, Ark. Math., **13** (1975), 161–207. [2](#)
- [10] Folland, G.B., & Stein, E.M., *Estimates for the  $\bar{\partial}_b$  Complex and Analysis on the Heisenberg Group*, Comm. Pure Appl. Math., **27** (1974), 429–522. [2](#)
- [11] Gamara, N., *The CR Yamabe conjecture the case  $n = 1$* , J. Eur. Math. Soc. (JEMS) **3** (2001), no. 2, 105–137. [3](#)
- [12] Gamara, N. & Yacoub, R., *CR Yamabe conjecture – the conformally flat case*, Pacific J. Math. **201** (2001), no. 1, 121–175. [3](#)
- [13] Garofalo, N. & Vassilev, D., *Symmetry properties of positive entire solutions of Yamabe type equations on groups of Heisenberg type*, Duke Math J, **106** (2001), no. 3, 411–449. [2](#)
- [14] Graham, C. R., & Lee, John M., *Einstein metrics with prescribed conformal infinity on the ball*. Adv. Math. **87** (1991), no. 2, 186–225. [2](#)
- [15] Ivanov, S., Minchev, I., & Vassilev, D., *Quaternionic contact Einstein structures and the quaternionic contact Yamabe problem*, Memoirs Am. Math. Soc., vol. 231, number 1086 (2014). [2](#), [3](#), [4](#), [5](#), [6](#), [7](#), [9](#), [10](#), [15](#)
- [16] Ivanov, S., Minchev, I., & Vassilev, D., *Extremals for the Sobolev inequality on the seven dimensional quaternionic Heisenberg group and the quaternionic contact Yamabe problem*, J. Eur. Math. Soc. (JEMS) **12** (2010), no. 4, 1041–1067. [2](#), [3](#)
- [17] Ivanov, S., Minchev, I., & Vassilev, D., *The optimal constant in the  $L^2$  Folland-Stein inequality on the quaternionic Heisenberg group*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) Vol. XI (2012), 635–662. [2](#), [3](#), [4](#), [15](#)
- [18] Ivanov, S., Minchev, I., & Vassilev, D., *The qc Yamabe problem on 3-Sasakian manifolds and the quaternionic Heisenberg group*, arXiv:1504.03142. [2](#), [3](#), [4](#), [15](#)
- [19] Ivanov, S., & Vassilev, D., *Conformal quaternionic contact curvature and the local sphere theorem*, J. Math. Pures Appl. **93** (2010), 277–307. [3](#), [4](#), [5](#), [6](#), [7](#), [9](#), [10](#)
- [20] Ivanov, S., & Vassilev, D., *Extremals for the Sobolev Inequality and the Quaternionic Contact Yamabe Problem*, Imperial College Press Lecture Notes, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011. [2](#), [3](#)
- [21] Ivanov, S., & Vassilev, D., *The Lichnerowicz and Obata first eigenvalue theorems and the Obata uniqueness result in the Yamabe problem on CR and quaternionic contact manifolds*, Nonlinear Analysis **126** (2015) 262–323. [2](#)
- [22] Jerison, D., Lee, J. M., *Intrinsic CR normal coordinates and the CR Yamabe problem*, J. Diff. Geom. **29** (1989), 303–343. [3](#), [16](#), [22](#), [29](#)
- [23] Kunkel, Chr., *Quaternionic contact normal coordinates*, preprint, arXiv:0807.0465 [math.DG]. [3](#), [4](#), [8](#), [9](#), [10](#), [11](#), [21](#)
- [24] LeBrun, C., *On complete quaternionic-Kähler manifolds*. Duke Math. J. **63** (1991), no. 3, 723–743. [1](#)
- [25] Lee, J.M., Parker, T., *The Yamabe problem*, Bull. Amer. Math. Soc. **17** (1987), 37–91. [3](#)
- [26] Mostow, G. D., *Strong rigidity of locally symmetric spaces*, Annals of Mathematics Studies, No. 78. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1973. v+195 pp. [1](#)
- [27] Pansu, P., *Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un*, Ann. of Math. (2) **129** (1989), no. 1, 1–60. [1](#)
- [28] Schoen, R., *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geom., **20** (1984), no. 2, 479–495. [3](#)
- [29] Tanaka, N., *A differential geometric study on strongly pseudo-convex manifolds*, Lectures in Mathematics, Department of Mathematics, Kyoto University, No. 9. Kinokuniya Book-Store Co., Ltd., Tokyo, 1975 [2](#)
- [30] Wang, W., *The Yamabe problem on quaternionic contact manifolds*, Ann. Mat. Pura Appl., **186** (2007), no. 2, 359–380. [2](#), [3](#), [4](#)
- [31] Webster, S. M., *Pseudo-hermitian structures on a real hypersurface*, J.Diff. Geom., **13** (1979), 25–41. [2](#)

(Stefan Ivanov) UNIVERSITY OF SOFIA, FACULTY OF MATHEMATICS AND INFORMATICS, BLVD. JAMES BOURCHIER 5, 1164, SOFIA, BULGARIA

AND INSTITUTE OF MATHEMATICS AND INFORMATICS, BULGARIAN ACADEMY OF SCIENCES

E-mail address: [ivanovsp@fmi.uni-sofia.bg](mailto:ivanovsp@fmi.uni-sofia.bg)

(Alexander Petkov) UNIVERSITY OF SOFIA, FACULTY OF MATHEMATICS AND INFORMATICS, BLVD. JAMES BOURCHIER 5, 1164, SOFIA, BULGARIA

E-mail address: [a\\_petkov\\_fmi@abv.bg](mailto:a_petkov_fmi@abv.bg)